Uniform Definability in Assertability Semantics[*]

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Abstract

This paper compares two notions of expressive power for a logical language and shows how they come apart. In particular, it introduces a simple framework called *assertability semantics* for handling puzzling features of the interaction of epistemic modals and disjunction. As a consequence of the solution to those puzzles, it is shown that the disjunction is in fact definable: every sentence is equivalent to a sentence without disjunction. But we then prove that the disjunction is not *uniformly* definable: no schematic definition of it can be given in terms of the other connectives of the fragment. We also consider the extension with inquisitive disjunction and prove that it is expressively complete.

As one of its benefits, logical semantics for natural language allows one to precisely answer questions about the *expressive power* of various fragments. Typically, one answers: what classes of structures can be defined by the fragment in question? Because, however, expressions have a given syntactic category and therefore semantic type, logical semanticists should be interested in more fine-grained conceptions of expressive power. In particular, which *operations* on the relevant classes of structures are definable? This paper shows how the two kinds of expressive power can come apart, by studying the interaction of disjunction and modals in the framework of assertability, or state-based, semantics. We show that although the so-called 'split' disjunction allows no new structures to be defined, its operation is not definable, i.e. the connective is not *uniformly* definable.

The paper is structured as follows. Section 1 presents some puzzling data on the behavior of epistemic modals and disjunction. Section 2 introduces a simple assertability semantics and develops some of its basic properties. Section 3 shows that disjunction is definable, while Section 4 introduces the concept of uniform definability and shows that disjunction is not uniformly definable. Section 5 explores the addition of inquisitive disjunction. We show that the resulting system is *expressively complete*, in that every set of states can be defined by a formula. Therefore, disjunction remains definable in this setting. It is conjectured that it still fails to be uniformly definable. Finally, we conclude by discussing future directions. We stress that the paper primarily aims to illustrate the contrast between the two forms of expressive power. The particular assertability semantics developed, while handling some data very elegantly, has empirical problems that are addressed in other work.[1]

1 Some Puzzles of Epistemic Modals and Disjunction

While the primary motivation of the present paper concerns two types of expressive power, the semantic system to be studied is designed to capture some very puzzling behavior of the interaction between (epistemic) modals and disjunctions, which will now be introduced.

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[1]See Steinert-Threlkeld [2017], chapter 3, "Pragmatic Expressivism and Non-Disjunctive Properties".

Our first puzzle concerns the well-known problem of free-choice possibility, in which certain disjunctions with possibility modals entail conjunctions of possibilities. In particular, in the epistemic case, such inferences appear to arise when disjunctions scope over modals.[2]

(1) Bernie Sanders might or might not win the Democratic nomination.
$\rightsquigarrow$ Bernie Sanders might win and he might not win.

(2) Maria might be at Science Park or she might be in the city center.
$\rightsquigarrow$ Maria might be at Science Park and she might be in the center.

These observations motivate (WFC): $\Diamond p \vee \Diamond q$ entails $\Diamond p \wedge \Diamond q$.

Our second puzzle concerns how the interpretation of modals is constrained by their linguistic context, including when embedded under disjunctions. Consider:

(3) Jennifer is at home and might be sick.

(3) is only felicitous if some sick-world is one in which Jennifer is at home. If a speaker knows that Jennifer can't be home sick, it is infelicitous. Dorr and Hawthorne [2013] observe that this 'inherited constraint' survives embedding under disjunction.

(4) Either Jennifer is at home and might be sick, or she's playing hookie.

Again, (4) can only be asserted when it's possible that Jennifer is sick and at home. This observation motivates (IC): $(p \wedge \Diamond q) \vee r$ entails $\Diamond(p \wedge q)$.

For the final puzzle, notice that sentences like (5) and their order variants – so-called *epistemic contradictions* – are notoriously marked.

(5) # It's raining and it might not be.

Moreover, as Yalcin [2007] and others have argued, the markedness survives embedding in a wide variety of contexts, including the antecedents of conditionals and under attitude verbs. This contrasts with the sentences which gloss 'might' as 'for all that the relevant group knows'.

(6) # If it's raining and it might not be, you should take the umbrella.

(7) # José thinks both that it might be raining and it isn't.

Similarly, but more puzzlingly, Mandelkern [2017][3] has observed that disjoining epistemic contradictions also sounds terrible. Suppose that you have a lottery ticket but don't yet know the outcome. Even in such a situation, (8) cannot be felicitously asserted.

(8) # Either I'll win and I might not, or I'll lose and I might not.

Again, the pattern is robust across order variations and different contexts. By contrast with the other embeddings, nearly no existing theory, including dynamic and domain semantics, can capture this infelicity. This discussion motivates (DEC): $(p \wedge \Diamond \neg p) \vee (q \wedge \Diamond \neg q)$ (and their variants) are inconsistent.

2 Assertability Semantics

To handle the phenomena just discussed, we will focus on the language $\mathcal{L}$ that contains proposition letters, negation $\neg$, conjunction $\wedge$, disjunction $\vee$, and a possibility modal $\Diamond$. We write

[2] See, among others, Zimmermann [2000], Geurts [2005], Aloni [2016].

[3] See chapter 1, "Bounded Modality".

$\mathcal{L}_P$ for the language of propositional logic, i.e. without $\Diamond$ and $\mathcal{L}^-$ for the language without $\vee$. Throughout, $\Box\varphi := \neg\Diamond\neg\varphi$.

We will call an *information model* a pair $M = \langle W, V\rangle$ of a set of possible worlds and a valuation V assigning subsets of W to proposition letters. Formulas of $\mathcal{L}$ will be interpreted at *information states* $\mathbf{s} \subseteq W$. We recursively define the relation $M, \mathbf{s} \Vdash \varphi$, to be read as "$\varphi$ is assertable relative to information $\mathbf{s}$". This is intended to capture the following: if an agent has the information $\mathbf{s}$ as her belief-state, then it is epistemically appropriate for her to assert φ.

Definition 1 (Hawke and Steinert-Threlkeld [2016])**.**

$\mathbf{s} \Vdash p$	iff	$\mathbf{s} \subseteq V(p)$
$\mathbf{s} \Vdash \neg\varphi$	iff	for every $w \in \mathbf{s}$, $\{w\} \nVdash \varphi$
$\mathbf{s} \Vdash \varphi \wedge \psi$	iff	$\mathbf{s} \Vdash \varphi$ and $\mathbf{s} \Vdash \psi$
$\mathbf{s} \Vdash \varphi \vee \psi$	iff	$\mathbf{s}_1 \Vdash \varphi$ and $\mathbf{s}_2 \Vdash \psi$ for some $\mathbf{s}_1, \mathbf{s}_2$ such that $\mathbf{s} = \mathbf{s}_1 \cup \mathbf{s}_2$
$\mathbf{s} \Vdash \Diamond\varphi$	iff	for some $w \in \mathbf{s}$, $\{w\} \Vdash \varphi$

We write $\Gamma \Vdash \varphi$ iff for every $M, \mathbf{s}$, if $\mathbf{s} \Vdash \gamma$ for every $\gamma \in \Gamma$, then $\mathbf{s} \Vdash \varphi$; and $\varphi \equiv \psi$ iff $\{\varphi\} \Vdash \psi$ and $\{\psi\} \Vdash \varphi$. Let $[\![\varphi]\!]_M = \{\mathbf{s} : M, \mathbf{s} \Vdash \varphi\}$ and $[\![\varphi]\!] = \{\langle M, \mathbf{s}\rangle : \mathbf{s} \in [\![\varphi]\!]_M\}$.

These clauses are rather intuitive: p is assertable relative to some information just in case that information leaves open only p worlds. $\neg\varphi$ is assertable only if the information leaves open no φ worlds. A disjunction is assertable just in case the information is *covered* by a piece of information corresponding to each disjunct.[4] And $\Diamond\varphi$ is assertable just in case a φ possibility is left open by the information.

We first observe that the non-modal fragment behaves classically: a sentence of propositional logic is assertable at a state just when it is clasically true at every world in that state. This quickly enables us to observe that this system satisfies (DEC). In the next section, we will additionally show that the system satisfies both (IC) and (WFC).

Fact 1. For every $\varphi \in \mathcal{L}_P$, (i) $\mathbf{s} \Vdash \varphi$ iff $\{w\} \Vdash \varphi$ for every $w \in \mathbf{s}$; (ii) $\{w\} \Vdash \varphi$ iff $v^*_w(\varphi) = 1$ where v^*_w is the classical propositional extension of the valuation given by $v_w(p) = 1$ iff $w \in V(p)$.

Fact 2. Epistemic contradictions are inconsistent: for every $\varphi \in \mathcal{L}_P, M, \mathbf{s}$: $\mathbf{s} \nVdash \varphi \wedge \Diamond\neg\varphi$.

The restriction to formulas without modals is both essential and justified by the literature, where all of the examples are of that type. In particular, the essential restriction is to formulas which are *flat*, in the sense of part (i) of Fact 1. To see that the restriction is essential, we note that $\Diamond p \wedge \Diamond\neg\Diamond p$ turns out equivalent to $\Diamond p \wedge \Diamond\neg p$, which holds at any information state with both a p and a $\neg p$ world. We find this prediction plausible, but leave its defense to future work.

Corollary 1. The assertability semantics satisfies (DEC): for all $\varphi, \psi \in \mathcal{L}_P$, and every $M, \mathbf{s}$:

$$\mathbf{s} \nVdash (\varphi \wedge \Diamond\neg\varphi) \vee (\psi \wedge \Diamond\neg\psi)$$

Proof. For the disjunction to be assertable at $\mathbf{s}$, there would have to be two sub-states whose union is $\mathbf{s}$, one at which each disjunct is assertable. By Fact 2, no such sub-states exist. □

Before proceeding, we record one important fact about the semantics and one definition, both of which will be used later. We use P as a variable for sets of proposition letters, and P_φ for the set of such letters occurring in φ.

[4] See Simons [2005] for more on (super-)covers and disjunctions. Observe that our definition does not require the sub-states for a disjunct to be non-empty. More on that later.

Proposition 1. *For all $\varphi \in \mathcal{L}$, if $M, \boldsymbol{s} \Vdash \varphi$ and $M', \boldsymbol{s}' \Vdash \varphi$, then $M \sqcup M', \boldsymbol{s} \cup \boldsymbol{s}' \Vdash \varphi$, where $M \sqcup M'$ is disjoint union of models, defined in the obvious way.*

Proof. By induction. We show the disjunction case, leaving the rest to the reader. Suppose $M, \mathbf{s} \Vdash \varphi \vee \psi$ and that $M', \mathbf{s}' \Vdash \varphi$. Then there are $\mathbf{s}_1, \mathbf{s}_2$ such that $\mathbf{s} = \mathbf{s}_1 \cup \mathbf{s}_2$, $\mathbf{s}_1 \Vdash \varphi$, and $\mathbf{s}_2 \Vdash \psi$. And *mutatis mutandis* for $\mathbf{s}'$. Then, by the inductive hypothesis, $\mathbf{s}_1 \cup \mathbf{s}_1' \Vdash \varphi$ and $\mathbf{s}_2 \cup \mathbf{s}_2' \Vdash \psi$. Because $\mathbf{s} \cup \mathbf{s}'$ is itself the union of these two unions, we have that $\mathbf{s} \cup \mathbf{s}' \Vdash \varphi \vee \psi$, as desired. □

Definition 2. $M^P := \langle \mathcal{P}(P), V^P \rangle$ where $V^P(p) = \{X \in \mathcal{P}(P) : p \in X\}$.

3 Definability

In this section, we show that disjunction is definable in terms of the other connectives in the following sense: for every formula in the language including disjunction ($\mathcal{L}$), there is a formula in the language without disjunction ($\mathcal{L}^-$) which is equivalent to it. The proof of this result uses a normal form theorem, which will be the main result of this section. The normal form result also yields immediate proofs that the system satisfies (IC) and (WFC). In the next section, we introduce the concept of *uniform definability* and prove that $\vee$ is not uniformly definable.

Our normal form will show that every formula is equivalent to one of the form $\varphi_b \wedge \Diamond\varphi_a^1 \wedge \cdots \wedge \Diamond\varphi_a^n$, where φ_b and all of the φ_a^i are modal-free formulas. This has an intuitive interpretation: every formula places two types of constraint on an information state: it must *entail* s certain piece of information and it must *be compatible with* certain others.If one thinks of the information state as an asserter's belief worlds, then an assertion of φ expresses a single belief and some *abeliefs*, where an agent abelieves p iff p is compatible with her beliefs. Whence the subscripts b and a. We now make this precise, showing how to simultaneously generate the formula to be believed and the set of formulas to be abelieved.

Definition 3. We simultaneously define two translations $(\cdot)_b : \mathcal{L} \to \mathcal{L}_P$ and $(\cdot)_a : \mathcal{L} \to \mathcal{P}(\mathcal{L}_P)$:

$$
\begin{aligned}
(p)_b &= p & (p)_a &= \varnothing \\
(\neg\varphi)_b &= \neg\left((\varphi)_b \wedge \bigwedge (\varphi)_a\right) & (\neg\varphi)_a &= \varnothing \\
(\varphi \wedge \psi)_b &= (\varphi)_b \wedge (\psi)_b & (\varphi \wedge \psi)_a &= (\varphi)_a \cup (\psi)_a \\
(\varphi \vee \psi)_b &= (\varphi)_b \vee (\psi)_b & (\varphi \vee \psi)_a &= \{(\varphi)_b \wedge \varphi_a : \varphi_a \in (\varphi)_a\} \cup \{(\psi)_b \wedge \psi_a : \psi_a \in (\psi)_a\} \\
(\Diamond\varphi)_b &= \top & (\Diamond\varphi)_a &= \{(\varphi)_b\} \cup (\varphi)_a
\end{aligned}
$$

So: p expresses belief in p. Similarly, $\Diamond\varphi$ expresses a trivial belief, but abelief in the belief expressed by φ as well as all of its abeliefs. $\varphi \wedge \psi$ expresses the belief in the conjunction of the beliefs expressed by φ and by ψ as well as all of the abeliefs expressed by each of φ and ψ. Note that $\neg\varphi$ always expresses only a belief; this resembles the fact that $\neg$ also removes inquisitivity in inquisitive semantics. The clause for disjunction will be illuminated in what follows.

Theorem 1 (Normal Form)**.** *For every $\varphi \in \mathcal{L}$, $\varphi \equiv (\varphi)_b \wedge \bigwedge \{\Diamond\varphi_a : \varphi_a \in (\varphi)_a\}$.*

Proof. By induction. We show only the disjunction case and leave the rest for the reader. So suppose that $\varphi \equiv (\varphi)_b \wedge \bigwedge \{\Diamond\varphi_a : \varphi_a \in (\varphi)_a\}$, and *mutatis mutandis* for ψ. Let $\mathbf{s} \Vdash \varphi \vee \psi$. Then there is an $\mathbf{s}_1 \Vdash (\varphi)_b \wedge \bigwedge \{\Diamond\varphi_a : \varphi_a \in (\varphi)_a\}$ and an $\mathbf{s}_2 \Vdash (\psi)_b \wedge \bigwedge \{\Diamond\psi_a : \psi_a \in (\psi)_a\}$ such that $\mathbf{s} = \mathbf{s}_1 \cup \mathbf{s}_2$. Now, by Fact 1, $(\varphi)_b$ holds at every world in $\mathbf{s}_1$. Since $\Diamond\varphi_a$ holds in $\mathbf{s}_1$, there is a φ_a world. Therefore, there is a $\varphi_a \wedge \varphi_b$ world in $\mathbf{s}_1$, which is also in $\mathbf{s}$. Therefore, $\mathbf{s} \Vdash \Diamond\left((\varphi)_b \wedge \varphi_a\right)$

for each φ_a. The same holds for ψ. Again by Fact 1, we also have that $\mathbf{s} \Vdash (\varphi)_b \vee (\psi)_b$, since every world satisfies one of the disjuncts from $\mathcal{L}_P$. So, $\mathbf{s} \Vdash (\varphi \vee \psi)_b \wedge \bigwedge \{\Diamond \chi_a : \chi_a \in (\varphi \vee \psi)_a\}$. The reader can verify that the reasoning above holds in reverse as well. □

Corollary 2. The assertability semantics satisfies (IC): for every $\varphi, \psi, \chi \in \mathcal{L}_P$,

$$(\varphi \wedge \Diamond \psi) \vee \chi \Vdash \Diamond(\varphi \wedge \psi)$$

Proof. Theorem 1 and Definition 3 yield that $(\varphi \wedge \Diamond \psi) \vee \chi$ is equivalent to $(\top \vee \chi) \wedge \Diamond(\varphi \wedge \psi)$, which clearly entails $\Diamond(\varphi \wedge \psi)$. □

Corollary 3. The assertability semantics satisfies (WFC): for every φ, ψ:

$$\Diamond \varphi \vee \Diamond \psi \equiv \Diamond \varphi \wedge \Diamond \psi$$

We observe two striking features of this equivalence. On the one hand, $p \vee q$ does not entail $\Diamond p \wedge \Diamond q$. This is because the sub-state required for p or for q is allowed to be empty.[5] But putting a modal under the disjunction forces each sub-state to be non-empty. This allows our system to preserve classical logic for the propositional fragment while still generating wide-scope free-choice inferences. On the other hand, we do not get narrow-scope free choice as an entailment. In other words: $\Diamond(p \vee q) \nVdash \Diamond p \wedge \Diamond q$. A counter-example: a state with a single $p \wedge \neg q$ world. While this might be seen as a problem, we observe that it can be shown that the standard (recursive) pragmatic explanation of narrow-scope free choice[6] can be re-created in this system. Whether this is a plausible combination of the interaction between scope and free-choice licensing will remain for future work.

Proposition 2. *For every $\varphi \in \mathcal{L}$, there is a $\varphi^* \in \mathcal{L}^-$ such that $\varphi \equiv \varphi^*$.*

Proof. By Theorem 1, $\varphi \equiv (\varphi)_b \wedge \bigwedge \{\Diamond \varphi_a : \varphi_a \in (\varphi)_a\}$. By the two parts of Fact 1, the propositional formulas $(\varphi)_b$ and all of the formulas φ_a can be replaced by disjunction-free formulas while preserving equivalence. □

In other words, Proposition 2 says that $\vee$ is definable in terms of $\{\neg, \wedge, \Diamond\}$. It turns out, however, that for given formulas φ, ψ with a disjunction, the equivalent formulas without the disjunction may bear no resemblance to each other. For example, $p \vee q$ is equivalent to $\neg(\neg p \wedge \neg q)$, while $\Diamond p \vee \Diamond q$ is equivalent to $\Diamond p \wedge \Diamond q$, but not to the corresponding De Morgan formula, since negation removes abeliefs. The question thus arises naturally: is this a defect of the De Morgan formulas, or is something deeper happening? In the next section, we answer that something deeper is going on.

4 Uniform Definability

Having thus shown that $\vee$ is definable, we turn to showing that there is no schematic definition of the connective. The precise concept behind the idea of a schematic definition is the following.

Definition 4 (Uniform Definability)**.** Let $\mathcal{L}_1, \mathcal{L}_2$ be two languages interpreted in the same class of models. An n-ary connective $*$ in $\mathcal{L}_1$ is *uniformly definable* in $\mathcal{L}_2$ iff there is a formula $\varphi_*[p_1, \ldots, p_n] \in \mathcal{L}_2$ such that for all $\psi_1, \ldots, \psi_n \in \mathcal{L}_2$, $*(\psi_1, \ldots, \psi_n) \equiv \varphi_*[p_1/\psi_1, \ldots, p_n/\psi_n]$.

[5] Here we depart from, among others, Aloni [2016]. We welcome this departure, but postpone a full discussion.

[6] See Kratzer and Shimoyama [2002], Fox [2007] among others.

The proof of the main result hinges on the way in which the various connectives interact with formulas that are *upward*-closed, in addition to those that are downward-closed. Note that the modal let us define such sets of sets, which are not definable in standard inquisitive semantics. In fact, this failure of downward-closure (also known as persistence) was also one of the key motivating features of the dynamic approach to epistemic modals.[7]

Definition 5. Let X be a set of sets of worlds. X is downward-closed – X is $\downarrow$ – iff if $s \in X$ and $t \subseteq s$, then $t \in X$. X is upward-closed – X is $\uparrow$ – iff if $s \in X$ and $s \subseteq t$, then $t \in X$. We say that X is netural – X is $\sim$ – iff X is neither $\downarrow$ nor $\uparrow$.

We will call $D := \{\uparrow, \downarrow, \sim\}$ the set of *directions* that a set of sets (or a formula) can have.Variables like d_i will range over this set. For a formula φ, we say that φ is d iff $[\![\varphi]\!]_M$ is d for every information model M. Our proof will show that $\vee$ interacts with upward-closed formulas differently than any formula defined without it. As an illustration, we note a simple fact about the disjunction.

Fact 3. If φ is $\uparrow$, then $\varphi \vee \psi$ is $\uparrow$.

Proof. Suppose φ is $\uparrow$ and let $\mathbf{s}$ be such that $\mathbf{s} \Vdash \varphi \vee \psi$ and let $\mathbf{t} \supseteq \mathbf{s}$. We have that $\mathbf{s}_1 \Vdash \varphi$ and $\mathbf{s}_2 \Vdash \psi$ for some $\mathbf{s}_1, \mathbf{s}_2$ such that $\mathbf{s} = \mathbf{s}_1 \cup \mathbf{s}_2$. Because φ is $\uparrow$, it follows that $\mathbf{s}_1 \cup (\mathbf{t} \setminus \mathbf{s}) \Vdash \varphi$. Since $\mathbf{s}_1 \cup (\mathbf{t} \setminus \mathbf{s}) \cup \mathbf{s}_2 = \mathbf{s} \cup \mathbf{t} \setminus \mathbf{s} = \mathbf{t}$, we have that $\mathbf{t} \Vdash \varphi \vee \psi$. □

Definition 6. Let $\varphi[p_1, \ldots, p_n]$ be a formula.

- φ is *d-enforcing* iff $\varphi[p_1/\psi_1, \ldots p_n/\psi_n]$ is d for every $\psi_1, \ldots, \psi_n$.
- φ is *d-promoting* iff $\varphi[p_1/\psi_1, \ldots, p_n/\psi_n]$ is d if some ψ_i is d.

We say an n-ary connective $*$ is d-enforcing (resp. d-promoting) if $*(p_1, \ldots, p_n)$ is. So, for example: $\Diamond$ is $\uparrow$-enforcing and $\vee$ is $\uparrow$-promoting (this is the content of Fact 3).

Our main result will concern the interaction of being $\uparrow$-promoting and $\uparrow$-enforcing. The key idea will be that it is only the disjunction that allows a formula to 'flip' from being downward-closed to upward-closed whenever an upward-closed formula is substituted. Special care will have to be taken to distinguish the disjunction from atoms, which are also (trivially) $\uparrow$-promoting. Before proceeding, we record a couple of helper facts.

Fact 4. φ is $\uparrow$ if and only if $\varphi_b \equiv \top$

Proof. $\Rightarrow$: suppose $\varphi_b \not\equiv \top$. Then there is a $w \in M^{P_{\varphi_b}}$ such that $\{w\} \nVdash \varphi_b$, i.e. $\{w\} \Vdash \neg\varphi_b$. Then, by construction, $M^{P_{\varphi_b}}, \mathcal{P}(P_{\varphi_b}) \Vdash \Diamond\neg\varphi_b$. So, by the epistemic contradiction Fact 2, $M^{P_{\varphi_b}}, \mathcal{P}(P_{\varphi_b}) \nVdash \varphi_b$. From this, it follows that φ_b is not $\uparrow$ since $\varnothing \Vdash \varphi_b$. The $\Leftarrow$ direction is immediate from Theorem 1. □

Fact 5. $\varphi \wedge \psi$ is $\uparrow$ if and only if φ is $\uparrow$ and ψ is $\uparrow$.

Proof. By Fact 4, $\varphi \wedge \psi$ is $\uparrow$ iff $(\varphi \wedge \psi)_b \equiv \top$. But, by Definition 3, we have $\varphi_b \wedge \psi_b \equiv \top$, which holds iff $\varphi_b \equiv \top$ and $\psi_b \equiv \top$ iff (again by Fact 4) φ is $\uparrow$ and ψ is $\uparrow$. □

Theorem 2. *Every formula $\varphi[p_1, \ldots, p_n]$ in $\mathcal{L}^-$ has the following property:*

(*) If φ is $\uparrow$-promoting, then φ is $\uparrow$-enforcing or uniformly equivalent to a proposition letter.

[7] As began by Veltman [1996] and further developed by many since then.

Proof. We proceed by induction on formulas. Proposition letters clearly satisfy (*), by making the consequent true. So assume that φ_1, φ_2 satisfy (*).

- $\neg\varphi_1$: because $\neg\varphi_1$ is $\downarrow$-enforcing, it is not $\uparrow$-promoting and so trivially satisfies (*).
- $\varphi_1 \wedge \varphi_2$: suppose that $\varphi_1 \wedge \varphi_2$ is $\uparrow$-promoting, i.e. for all $\psi_1, \ldots, \psi_n$, if some ψ_i is $\uparrow$, then $\varphi_1 \wedge \varphi_2[p_1/\psi_1, \ldots, p_n/\psi_n]$ is $\uparrow$. By Fact 5, both $\varphi_1[p_1/\psi_1, \ldots, p_n/\psi_n]$ and $\varphi_2[p_1/\psi_1, \ldots, p_n/\psi_n]$ are also $\uparrow$. In other words, φ_1 and φ_2 are also $\uparrow$-promoting. By the inductive hypothesis, each one is thus either $\uparrow$-enforcing or uniformly equivalent to a proposition letter.

  If both φ_1 and φ_2 are $\uparrow$-enforcing, then so too is $\varphi_1 \wedge \varphi_2$, again by Fact 5.

  If φ_1 and φ_2 are uniformly equivalent to the *same* proposition letter – say, p_i – then we also have $\varphi_1 \wedge \varphi_2$ is uniformly equivalent to p_i.

  The only remaining case is when one conjunct – say, φ_1 – is uniformly equivalent to a proposition letter (without loss of generality, suppose it is p_1), and φ_2 is not uniformly equivalent to that same proposition letter. In this case, we have that $\varphi_1 \wedge \varphi_2$ is not uniformly equivalent to a proposition letter. But, for any $\uparrow$ formula ψ_n, we have that $\varphi_1 \wedge \varphi_2[p_1/p_1, \ldots, p_n/\psi_n]$ is equivalent to $p_1 \wedge \varphi_2[p_1/p_1, \ldots, p_n/\psi_n]$. But this entails p_1, and so is not $\uparrow$. Thus, $\varphi_1 \wedge \varphi_2$ is not $\uparrow$-promoting, contradicting the assumption.

  We have thus shown that $\varphi_1 \wedge \varphi_2$ satisfies (*).
- $\Diamond\varphi_1$: because $\Diamond\varphi_1$ is $\uparrow$-enforcing, it satisfies (*). □

Theorem 3. $\vee$ *is not uniformly definable in* $\mathcal{L}^{-}$.

Proof. $p \vee q$ is $\uparrow$-promoting, but neither $\uparrow$-enforcing nor uniformly equivalent to a proposition letter. Therefore, the previous Theorem shows that no formula can uniformly define it. □

5 Adding Inquisitive Disjunction

Because the language $\mathcal{L}^{-}$ is relatively weak, it is worth investigating richer fragments. We now consider the language $\mathcal{L}_{\mathbb{W}}$, which adds a new symbol $\mathbb{W}$ for inquisitive disjunction. We augment Definition 1 with the clause from inquisitive semantics:

$$\mathbf{s} \Vdash \varphi \mathbin{\mathbb{W}} \psi \quad \text{iff} \quad \mathbf{s} \Vdash \varphi \text{ or } \mathbf{s} \Vdash \psi$$

In this section, we show that $\vee$ is once again definable, but must only conjecture that it is not uniformly definable. The definability result goes as before: we prove a normal form result which finds, for every formula, an equivalent normal form in which $\vee$ does not occur. The normal form has a pleasant character: every formula is equivalent to an inquisitive disjunction of normal forms in the sense of Theorem 1.

Fact 6. $\neg(\varphi \mathbin{\mathbb{W}} \psi) \equiv \neg\varphi \wedge \neg\psi$

Proof. $\mathbf{s} \Vdash \neg(\varphi \mathbin{\mathbb{W}} \psi)$ iff $\{w\} \nVdash \varphi \mathbin{\mathbb{W}} \psi$ for all $w \in \mathbf{s}$. And $\{w\} \nVdash \varphi \mathbin{\mathbb{W}} \psi$ iff $\{w\} \nVdash \varphi$ and $\{w\} \nVdash \psi$. This latter condition holds for every $w \in \mathbf{s}$ iff $\mathbf{s} \Vdash \neg\varphi \wedge \neg\psi$. □

Theorem 4 (Normal Form for $\mathcal{L}_{\mathbb{W}}$)**.** *Every formula* $\varphi \in \mathcal{L}_{\mathbb{W}}$ *is equivalent to a formula* φ^* *of the form*

$$\mathop{\mathbb{W}}_i \varphi_0^i \wedge \bigwedge_j \Diamond\varphi_j^i$$

where all of the $\varphi_j^i \in \mathcal{L}_P$.

Proof. The base case – atoms – is trivial; so too is the inquisitive disjunction case. Using the inductive hypothesis that each subformula has a normal form, we handle the rest of the connectives as follows.

- $\neg\varphi$: $\neg\left(\bigvee\!\!\!\bigvee_i \varphi_0^i \wedge \bigwedge_j \Diamond\varphi_j^i\right)$ is equivalent, by Fact 6, to $\bigwedge_i \neg\left(\varphi_0^i \wedge \bigwedge_j \Diamond\varphi_j^i\right)$. Using Theorem 1 and the corresponding Definition 3, this is equivalent to $\bigwedge_i \neg\left(\varphi_0^i \wedge \bigwedge_j \varphi_j^i\right)$, of the desired form.

- $\varphi \wedge \psi$: note that $\left(\bigvee\!\!\!\bigvee_i \varphi_0^i \wedge \bigwedge_j \Diamond\varphi_j^i\right) \wedge \left(\bigvee\!\!\!\bigvee_k \psi_0^k \wedge \bigwedge_\ell \Diamond\psi_\ell^k\right)$ is equivalent to $\bigvee\!\!\!\bigvee_{i,k} \varphi_0^i \wedge \bigwedge_j \Diamond\varphi_j^i \wedge \psi_0^k \wedge \bigwedge_\ell \Diamond\psi_\ell^k$

- $\varphi \vee \psi$: we have that $\left(\bigvee\!\!\!\bigvee_i \varphi_0^i \wedge \bigwedge_j \Diamond\varphi_j^i\right) \vee \left(\bigvee\!\!\!\bigvee_k \psi_0^k \wedge \bigwedge_\ell \Diamond\psi_\ell^k\right)$ holds at information state iff it has two sub-states whose union is the whole state, one of which satisfies the left inquisitive disjunction, one of which satisfies the right inquisitive disjunction. The reader can verify that this is therefore equivalent to $\bigvee\!\!\!\bigvee_{i,k}\left(\varphi_0^i \wedge \bigwedge_j \Diamond\varphi_j^i\right) \vee \left(\psi_0^k \wedge \bigwedge_\ell \Diamond\psi_\ell^k\right)$ We can then apply Theorem 1 to each inquisitive disjunct, to get an equivalent formula without the $\vee$ scoping over conjunctions, as desired.

- $\Diamond\varphi$: note that $\Diamond$ distributes over inquisitive disjunction, so that $\Diamond\left(\bigvee\!\!\!\bigvee_i \varphi_0^i \wedge \bigwedge_j \Diamond\varphi_j^i\right)$ is equivalent to $\bigvee\!\!\!\bigvee_i \Diamond\left(\varphi_0^i \wedge \bigwedge_j \Diamond\varphi_j^i\right)$ which is equivalent to $\bigvee\!\!\!\bigvee_i \Diamond\left(\varphi_0^i \wedge \bigwedge_j \varphi_j^i\right)$ by Theorem 1. The latter is of the desired form. $\square$

Corollary 4. $\vee$ is definable in $\mathcal{L}_{\mathbb{W}}^-$.

Proof. As before, $\vee$ can be removed from the $\mathcal{L}_P$ formulas while preserving equivalence. $\square$

Conjecture 1. *$\vee$ is not uniformly definable in $\mathcal{L}_{\mathbb{W}}^-$.*[8]

We note that $\mathcal{L}_{\mathbb{W}}$ is genuinely expressively richer, because inquisitive disjunctions are not closed under unions. In fact, we go on to show that $\mathcal{L}_{\mathbb{W}}$ is in a precise sense *maximally* expressive.

Proposition 3. *$\mathbb{W}$ is not definable in $\mathcal{L}$.* A fortiori*, it is not uniformly definable.*

Proof. $p \mathbin{\mathbb{W}} q$ is not closed under unions: in $M^{\{p,q\}}$, $\{p\} \Vdash p$, and $\{q\} \Vdash q$ (and so each supports $p \mathbin{\mathbb{W}} q$ as well), but $\{p,q\} \nVdash p \mathbin{\mathbb{W}} q$. By Proposition 1, no formula in $\mathcal{L}$ is equivalent to $p \mathbin{\mathbb{W}} q$. $\square$

Our notion of expressive completeness will run as follows: a language is complete if, for any finite set of proposition letters, any set of sets built out of those atoms can be defined by a formula. To make this precise, we introduce the notion of restricting a model to a set of proposition letters. For a set of letters P, $M\restriction_P$ will identify all worlds in W that agree on all of the proposition letters in P. In that sense, $M\restriction_P$ will contain all and only what M can see concerning the atoms in P.

Definition 7. Let M be an information model and P a set of proposition letters. The *restriction of M to P* – also called the P-reduct of M – is the model $M\restriction_P = \langle W/\equiv_P, V_P\rangle$ where $w \equiv_P w'$ iff for every $p \in P$, $w \in V(p)$ iff $w' \in V(p)$ and $V_P(p) = \left\{[w]_{\equiv_P} : w \in V(p)\right\}$. For $\mathbf{s} \subseteq W$, we define $\mathbf{s}\restriction_P := \left\{[w]_{\equiv_P} : w \in \mathbf{s}\right\}$. For $X \subseteq \mathcal{P}(W)$, we define $X\restriction_P := \{\mathbf{s}\restriction P : \mathbf{s} \in X\}$.

Definition 8. A language $\mathcal{L}$ is *expressively complete* iff: for every finite set of proposition letters P, M, $X \subseteq \mathcal{P}(W)$, there is a formula $\varphi \in \mathcal{L}$ such that $[\![\varphi]\!]_M\restriction_P = X\restriction_P$.

[8] Compare p. 163 of Ciardelli [2016], where he conjectures that $\vee$ is not uniformly definable in the system InqB, which lacks the modal but has a conditional.

Theorem 5. *$\mathcal{L}_{\mathbb{W}}$ is expressively complete.*

Proof. Let P be a finite set of proposition letters and $\mathbf{s}$ a set of worlds. For w, write $p_w := p$ if $w \in V(p)$ and $\neg p$ otherwise. Let $\varphi_w := \bigwedge_{p \in P} p_w$. Note that $w \equiv_P w'$ if and only if $\varphi_w = \varphi_{w'}$. Observe that $|\mathbf{s}\restriction_P| \leq 2^{|P|}$ which, since P is finite, is finite. The reader can verify that $[\![\bigvee_{[w] \in \mathbf{s}\restriction_P} \varphi_w \wedge \bigwedge_{[w] \in \mathbf{s}\restriction_P} \Diamond \varphi_w]\!]\restriction_P = \{\mathbf{s}\restriction_P\}$ i.e. that the formula defines exactly the set $\mathbf{s}\restriction_P$. We call the formula in brackets above $\varphi_{\mathbf{s}}$. Now, let X be a set of sets of worlds. As before, X must be finite. We then have that $[\![\mathbb{W}_{\mathbf{s} \in X\restriction_P} \varphi_{\mathbf{s}}]\!]\restriction_P = X\restriction_P$ which is of the desired form. □

This result is quite remarkable. It is known that both inquisitive logic and propositional dependence logic are expressively complete for *downward-closed* sets of sets.[9] The above result shows that adding a possibility modal – in a sense the simplest kind of upward-closed formula – to inquisitive semantics turns it into an expressively complete language. To put the point in a slogan: *a little bit of modality goes a long way.*

6 Conclusion

We introduced an assertability semantics to account for some puzzling data concerning the interaction of epistemic modals and disjunction. We proved that even though the disjunction is definable, it is not uniformly definable. This result shows that the standard conception of expressive power does not capture all that is of interest for a natural language semanticist. Even if a formal language can express the truth-conditions for every sentence in a fragment of interest, it can still fail to define all the operations denoted by functional vocabulary of the relevant fragment. Finally, we showed that the disjunction remains definable in the presence of inquisitive disjunction, and conjectured that it is still not uniformly definable. We also showed that adding a modal to inquisitive semantics renders it expressively complete.

Much work remains to be done. First, one would like to settle **Conjecture 1**. $\mathcal{L}_{\mathbb{W}}$ would thus be an expressively complete language which fails to uniformly define a natural connective. Secondly, one could investigate the disjunction in the setting of weak negation: $\mathbf{s} \Vdash -\varphi$ iff $\mathbf{s} \nVdash \varphi$, as first studied in Punčochář [2015]. It can be shown that both $\mathbb{W}$ and $\Diamond$ are uniformly definable (by $-(-p \wedge -q)$ and $-\neg\neg-p$, respectively) and that $\{-, \wedge, \neg\}$ is expressively complete. Nevertheless, we introduce two more conjectures.

Conjecture 2. *$\vee$ is not uniformly definable in $\{-, \wedge, \neg\}$.*

Conjecture 3. *$-$ is not uniformly definable in $\mathcal{L}_{\mathbb{W}}$.*

The latter seems especially plausible, given that $--\varphi \equiv \varphi$ for every φ. Furthermore, does the uniform definability result extend to more complicated semantic settings? For example, Aloni [2016], Steinert-Threlkeld [2017], Roelofsen [2017] all move to a *bilateral* setting – simultaneously defining what we would call assertability and deniability conditions – in order to to solve certain problems.[10] Extending the results of this paper to that setting will be non-trivial. Finally, a thorough investigation of the split disjunction (or close analogues) in both larger fragments and other frameworks would be fruitful.

[9] See Ciardelli [2009] for the former and Yang and Väänänen [2017] for the latter. In particular, $\{\neg, \mathbb{W}\}$ is expressively complete for downward-closed sets. Yang [2017] shows that $\mathbb{W}$ is not uniformly definable in dependence logic.

[10] In the system here, $\neg\neg\Diamond p \equiv p \not\equiv \Diamond p$, which has the consequence that $\neg\Box p \equiv \neg p \not\equiv \Diamond\neg p$. Bilateral systems can fix this defect, among others.

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