Nonmonotone Inductive Definitions

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Inductive Definitions as Sets of Clauses

Definition

A set $\mathcal{B}$ of clauses (of the form $A \Rightarrow b$) is an *inductive definition*.
A set X is $\mathcal{B}$-closed if $A \Rightarrow b \in \mathcal{B}$ and $A \subseteq X$ imply $b \in X$.
$I(\mathcal{B}) = \bigcap \{X \mid X \text{ is } \mathcal{B}\text{-closed}\}$ is *inductively generated* by $\mathcal{B}$.

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Example

A few motivating examples:

- $\mathbb{N} = I(\{\emptyset \Rightarrow 0\} \cup \{\{n\} \Rightarrow n+1\})$
- $WFF = I(\{\emptyset \Rightarrow p \mid p \in PV\} \cup \{\{\varphi\} \Rightarrow \neg\varphi\} \cup \{\{\varphi, \psi\} \Rightarrow \varphi \odot \psi \mid \odot \in \{\wedge, \vee, \rightarrow\}\})$
- Kleene's $\mathcal{O} = I(\{\emptyset \Rightarrow 0\} \cup \{\{a\} \Rightarrow 2^a\} \cup \{\{[e](n)\} \Rightarrow 3 \cdot 5^e\})$

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Note: what we generally refer to as "base cases" are clauses $\emptyset \Rightarrow b$.

Inductive Definitions as Monotone Operators

A well-motivated generalization of the previous definition:

Definition

An *inductive definition* on a set A is a monotone operator $\Phi : \mathcal{P}(A^n) \to \mathcal{P}(A^n)$

A set $S \subseteq \mathcal{P}(A^n)$ is Φ-*closed* iff $\Phi(S) \subseteq S$.

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Definition

The α-th stage of an inductive definition Φ is defined by transfinite recursion:

$$\Phi^\alpha := \Phi(\Phi^{<\alpha})$$

where

$$\Phi^{<\alpha} := \bigcup_{\xi<\alpha} \Phi^\xi$$

In other words, $\Phi^0 = \Phi(\emptyset)$, $\Phi^1 = \Phi(\Phi(0))$, ...

Fixed Points of Monotone Inductive Definitions

Recall an important lemma and its proof:

Lemma (Lemma 6.3.2)

Let Φ be an inductive definition on a set A. Then there is an ordinal $\sigma < \mathrm{card}(A)^+$ such that $\Phi^{<\sigma} = \Phi^\sigma$.

Proof.

By definition and the monotonicity of Φ,

$$\xi < \eta \Rightarrow \Phi^\xi \subseteq \Phi^\eta$$

Now, every $\Phi^\xi \subseteq A$, so $\mathrm{card}(\Phi^\xi) \leq \mathrm{card}(A)$. There are $\mathrm{card}(A)^+$ many ordinals below $\mathrm{card}(A)^+$. Therefore, if we had strict subset above for every ordinal $\leq \mathrm{card}(A)$, then there would be a $\xi < \mathrm{card}(A)^+$ with $\mathrm{card}(\Phi^\xi) > \mathrm{card}(A)$. Therefore, there is an ordinal $\sigma < \mathrm{card}(A)^+$ such that $\Phi^{<\sigma} = \Phi^\sigma$, i.e. σ is a fixed point. □

Monotonicity Too Strong

In this cardinality argument, we do not in fact need that Φ is a *monotone* operator. Since we define the hierarchy of stages on ordinals, which are transitive sets, it suffices simply for Φ to be *inflationary*, i.e. to satisfy $X \subseteq \Phi(X)$.

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In fact, what we will do is:

1. Drop monotonicity condition from definition.
2. Revise definition of stages to induce an inflationary operator.

The Plan

In this talk (following [Poh09, ch. 13] and [Poh08]), I will:

1. Briefly introduce nonmonotone inductive definitions.
2. Introduce prewellorderings and analyze their relationship to nonmonotone inductive definitions.
3. Describe a theory $(\Pi^0_1 - \mathrm{FXP})_0$ axiomatizing the existence of fixpoints for all Π^0_1-definable operators.
4. Prove that $||\mathrm{ID}_1|| \leq ||(\Pi^0_1 - \mathrm{FXP})_0||$.
5. Sketch an outline of the proof that $||(\Pi^0_1 - \mathrm{FXP})_0|| \leq ||\mathrm{ID}_1||$.

Stages of Nonmonotone Induction

Definition

Let $\Phi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ be an operator. We define the hierarchy of stages as

$$\Phi^{\alpha} := \Phi^{<\alpha} \cup \Phi(\Phi^{<\alpha})$$

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Definition

We call

$$\Phi^{<\infty} := \bigcup_{\xi \in On} \Phi^{\xi}$$

the *fixed-point generated by* Φ.

Examples of Nonmonotone Inductive Definitions

Example

Let Φ_0 and Φ_1 be two monotone operators (on $\mathcal{P}(\mathbb{N})$). Define

$$[\Phi_0, \Phi_1] = \{x \in \mathbb{N} \mid x \in \Phi_0(X) \vee (\Phi_0(X) \subseteq X \wedge x \in \Phi_1(X))\}$$

In other words, we iterate Φ_0 until a fixed point is reached and then iterate Φ_1 once, after which we keep repeating this process.

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In other words, we iterate Φ_0 until a fixed point is reached and then iterate Φ_1 once, after which we keep repeating this process. A particularly widely studied group of nonmonotone i.d.'s are

$$[\mathcal{F}_1, \ldots, \mathcal{F}_n] := \{[\Phi_1, \ldots \Phi_n] \mid \Phi_i \text{ is positively } \mathcal{F}_i\text{-definable}\}$$

See [RA74] for more details. (Also [Poh09, p. 335] for generalization of Kleene's $\mathcal{O}$.)

Fixed Point Lemma

With our modified definition of the hierarchy of stages, the fixed-point lemma holds in the nonmonotone case:

Lemma

Let $\Phi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ be an operator. Then there is an ordinal $\sigma < \omega_1$ such that $\Phi^{<\sigma} = \Phi^{\sigma}$. Moreover, it also holds for this σ that $\Phi^{<\infty} = \Phi^{<\sigma} = \Phi^{\sigma}$.

Proof.

Note that $\omega_1 = \mathrm{card}(\mathbb{N})^+$. The exact same argument as before yields a $\sigma < \omega_1$ that is a fixpoint of Φ. Clearly, $\Phi^{\sigma} \subseteq \Phi^{<\infty}$. We prove by induction that $\sigma \leq \tau \Rightarrow \Phi^{\tau} = \Phi^{<\sigma}$. Trivial if $\sigma = \tau$. If $\sigma < \tau$, the IH tells us $\Phi^{<\tau} = \Phi^{<\sigma}$, from which we have

$$\Phi^{\tau} = \Phi^{<\tau} \cup \Phi(\Phi^{<\tau}) = \Phi^{<\sigma} \cup \Phi(\Phi^{<\sigma}) = \Phi^{\sigma} = \Phi^{<\sigma}$$

$\square$

Some Norms

Some norms, as in the monotone case:

Definition

The *closure ordinal of* Φ is defined as $|\Phi| = \min\{\sigma \mid \Phi^\sigma = \Phi^{<\sigma}\}$.
The *inductive norm induced by* Φ, $|\cdot|_\Phi : \mathbb{N} \to On$, is given by

$$|n|_\Phi = \begin{cases} \min\{\alpha \mid n \in \Phi^\alpha\} & x \in \Phi^{<\infty} \\ \infty & \text{otherwise} \end{cases}$$

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A few simple results:

Lemma

$\Phi^\infty = \Phi^{|\Phi|} = \Phi^{<\infty}$.
$|\Phi| = \{|x|_\Phi \mid x \in \Phi^{<\infty}\}$ *In other words,* $|\cdot|_\Phi : \Phi^{<\infty} \twoheadrightarrow |\Phi|$.

Setting the Stage

Our goal: axiomatize Π^0_1-definable (nonmonotone) inductive definitions. Unlike in the case of monotone inductive definitions, we can't just proceed (as in ID_1) by axiomatizing "the least Φ-closed set". This is because the smallest fixed-point of a nonmonotone inductive definition may be an ordinal not available in $\mathcal{L}_{NT}$.
What we will axiomatize is properties of the *stage comparison* relations

$$x \preceq_\Phi y := \exists\alpha(x \in \Phi^\alpha \wedge y \notin \Phi^{<\alpha})$$
$$x \prec_\Phi y := \exists\alpha(x \in \Phi^\alpha \wedge y \notin \Phi^\alpha)$$

which we will prove are satisfied only by the appropriate fixed points.

Facts About Stage Comparison

Lemma

$$x \preceq_\Phi y \Leftrightarrow x \in \Phi^{<\infty} \wedge (y \in \Phi^{<\infty} \rightarrow y \not\prec_\Phi x)$$
$$x \prec_\Phi y \Leftrightarrow x \in \Phi^{<\infty} \wedge (y \in \Phi^{<\infty} \rightarrow y \not\preceq_\Phi x)$$

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Proof.

Only the $\Rightarrow$ direction of the first. Let $x \preceq_\Phi y$. For some α, $x \in \Phi^\alpha \wedge y \notin \Phi^{<\alpha}$. Therefore, $x \in \Phi^{<\infty}$. If $y \notin \Phi^{<\infty}$, done. If so, there is $\alpha \leq \beta$ with $y \in \Phi^\beta$. But also $x \in \Phi^\beta$, so $y \not\prec_\Phi x$.
The others are similar and straightforward. $\square$

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The others are similar and straightforward. □

Lemma

$\Phi^{|y|_\Phi} = \{x \mid x \preceq_\Phi y\}$
$\Phi^{<|y|_\Phi} = \{x \mid x \prec_\Phi y\}$
$x \preceq_\Phi y \Leftrightarrow x \prec_\Phi y \vee x \in \Phi(\{z \mid z \prec_\Phi y\})$.

Prewellorderings

Definition

A *prewellordering* is a transitive, total, well-founded binary relation.

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We will develop this notion in more detail via *norms*

$$f : P \twoheadrightarrow \lambda \in On$$

Our goal will be to find conditions that are uniquely satisfied by the stage comparison relations. These, then, will be axiomatized into $(\Pi^0_1 - \mathrm{FXP})_0$.

Prewellorderings Defined

Definition

Let P be a set and $f : P \twoheadrightarrow \lambda$ be a norm. The triple $(P, \preceq, \prec)$ is a *prewellordering* if

$$x \preceq y \Leftrightarrow x \in P \wedge (y \in P \Rightarrow f(x) \leq f(y))$$
$$x \prec y \Leftrightarrow x \in P \wedge (y \in P \Rightarrow f(x) < f(y))$$

Note that every norm on a set *induces* a prewellordering if we take the above biconditionals as definitions.

Prewellordering Theorem I

Theorem

$(\Phi^{<\infty}, \preceq_\Phi, \prec_\Phi)$ *is the unique prewellordering which satisfies*

$$x \preceq_\Phi y \Leftrightarrow x \prec_\Phi y \vee x \in \Phi(\{z \mid z \prec_\Phi y\}) \qquad (FP_\Phi)$$

Prewellordering Theorem II

Proof.

From the facts about stage comparison, we have that $|\cdot|_\Phi$ is a norm on $\Phi^{<\infty}$ whose induced prewellordering exactly is $(\preceq_\Phi, \prec_\Phi)$. These previous facts also show that (FP_Φ) is satisfied.
For uniqueness, let $(P, \preceq, \prec)$ be a prewellordering satisfying (FP_Φ). One can show by induction that

$$y \in P \Rightarrow \Phi^{f(y)} = \{z \mid z \preceq y\}$$
$$y \in P \Rightarrow f(y) = |y|_\Phi$$
$$\Phi^{|x|_\Phi} = \{z \mid z \preceq x\}$$

The first and third of these entail that $P = \Phi^{<\infty}$. Since the associated norm of P is $|\cdot|_\Phi$, we are done. $\square$

Eliminating Reference to Ordinals

In order to provide a sufficient axiomatization in $\mathcal{L}_{NT}$, we must find sufficient conditions for prewellordering that do not make reference to large ordinals via norms.

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Theorem

Let $P \subseteq \mathbb{N}$ and $(\preceq, \prec)$ be transitive relations. Then $(P, \preceq, \prec)$ is a prewellordering iff

$$x \preceq y \Leftrightarrow x \in P \wedge (y \in P \Rightarrow y \nprec x) \tag{PWO1}$$

$$x \prec y \Leftrightarrow x \preceq y \wedge y \npreceq x \tag{PWO2}$$

$$\prec \ \text{ is well-founded} \tag{PWO3}$$

Eliminating Reference to Ordinals

Proof.

$\Rightarrow$: Assume $(P, \preceq, \prec)$ is a pwo with norm f. Because f is onto an ordinal, $\prec$ is well-founded. We then have:

$$\begin{aligned}
x \preceq y &\Leftrightarrow x \in P \wedge (y \in P \Rightarrow f(x) \leq f(y)) \\
&\Leftrightarrow x \in P \wedge (y \in P \Rightarrow f(y) \not< f(x)) \\
&\Leftrightarrow x \in P \wedge (y \in P \Rightarrow x \not\prec y) \qquad \text{(PWO1)} \\
x \prec y &\Leftrightarrow x \in P \wedge (y \in P \Rightarrow f(x) < f(y)) \\
&\Leftrightarrow x \in P \wedge (y \in P \Rightarrow f(x) \leq f(y)) \\
&\quad \wedge (y \in P \Rightarrow (x \in P \wedge f(x) < f(y))) \\
&\Leftrightarrow x \preceq y \wedge y \not\preceq x \qquad \text{(PWO2)}
\end{aligned}$$

Eliminating Reference to Ordinals

Proof.

$\Rightarrow$: Assume $(P, \preceq, \prec)$ is a pwo with norm f. Because f is onto an ordinal, $\prec$ is well-founded. We then have:

$$
\begin{aligned}
x \preceq y &\Leftrightarrow x \in P \wedge (y \in P \Rightarrow f(x) \leq f(y)) \\
&\Leftrightarrow x \in P \wedge (y \in P \Rightarrow f(y) \nless f(x)) \\
&\Leftrightarrow x \in P \wedge (y \in P \Rightarrow x \nprec y) && \text{(PWO1)} \\
x \prec y &\Leftrightarrow x \in P \wedge (y \in P \Rightarrow f(x) < f(y)) \\
&\Leftrightarrow x \in P \wedge (y \in P \Rightarrow f(x) \leq f(y)) \\
&\qquad \wedge (y \in P \Rightarrow (x \in P \wedge f(x) < f(y))) \\
&\Leftrightarrow x \preceq y \wedge y \npreceq x && \text{(PWO2)}
\end{aligned}
$$

$\Leftarrow$: Define $f(x) := \mathrm{otyp}_{\prec}(x)$ and verify the original two conditions to be a prewellordering. $\square$

Further Characterizations

Definition

$D_{\preceq} := \{x \mid x \preceq x\}$ is called *the diagonalization of* $\preceq$.

Lemma

If $(P, \preceq, \prec)$ *satisfies (PWO1)-(PWO3), then* $P = D_{\preceq}$.

Lemma

If $(\preceq, \prec)$ *are transitive and satisfy (PWO1)-(PWO3), then*

$$x \prec y \preceq z \Rightarrow x \prec z$$
$$x \preceq y \prec z \Rightarrow x \prec z$$

Definable Operators

Theorem

Let Φ_F be definable, i.e. $\Phi_F(X) = \{x \in \mathbb{N} \mid F(X, x)\}$ and $(\preceq_F, \prec_F)$ be transitive relations satisfying (PWO1)-(PWO3) and

$$x \preceq_F y \Leftrightarrow x \prec_F y \vee F(\{z \mid z \prec_F y\}, x) \tag{FIX}$$

Then $(D_{\preceq_F}, \preceq_F, \prec_F)$ is a prewellordering, whence $D_{\preceq_F} = \Phi_F^{<\infty}$.

Proof.

Trivially a prewellordering. The second half follows from the Prewellordering Theorem. □

Basic Setup

In light of the previous theorem, (PWO1)-(PWO3) and (FIX) are the statements we need to axiomatize in order to express the existence of fixed-points for definable operators (which we will restrict to Π^0_1-definable).
We work in the language of second-order arithmetic. First, we will introduce a number of syntactic abbreviations.

$$Pair(x) :\Leftrightarrow Seq(x) \wedge lh(x) = 2$$
$$Rel(X) :\Leftrightarrow \forall x \in X\ Pair(x)$$

More Abbreviations

We introduce the x-slice of a relation:

$$X_x = \{y \mid \langle x, y\rangle \in X\}$$

and define that X codes two transitive orderings:

$$\begin{aligned} \mathit{Trans}(X) &:\Leftrightarrow \mathit{Rel}(X) \wedge \forall xyz(Xxy \wedge Xyz \to Xxz) \\ \mathit{PO}(X) &:\Leftrightarrow \forall x(x \in X \leftrightarrow \mathit{Pair}(x) \wedge ((x)_0 = 0 \vee (x)_0 = 1) \\ &\qquad \wedge \mathit{Trans}(X_0) \wedge \mathit{Trans}(X_1)) \end{aligned}$$

If $\mathit{PO}(X)$ holds, we write $\preceq_X := X_0$ and $\prec_X := X_1$.

(PWO1) and (PWO2) Formalized

To express (PWO1) and (PWO2) – in other words, that $(D_{\preceq_X}, \preceq_X, \prec_X)$ is a preordering – we define:

$$\begin{aligned} PRO(X) :\Leftrightarrow PO(X) \wedge \forall x \forall y (& \\ & [x \preceq_X y \leftrightarrow x \preceq_X x \wedge (y \preceq_X y \rightarrow y \not\prec_X x)] && \text{(PWO1)} \\ & \wedge [x \prec_X y \leftrightarrow x \preceq_X y \wedge y \not\preceq_X x]) && \text{(PWO2)} \end{aligned}$$

(PWO3) Formalized

We can write X is well-founded as:

$$WF(X) :\Leftrightarrow \forall Y(Y \subseteq field(X) \wedge \exists x(x \in Y) \rightarrow \exists x(x \in Y \wedge \forall y(Xyx \rightarrow y \notin Y)))$$

(PWO3) Formalized

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We can then say $(D_{\preceq_X}, \preceq_X, \prec_X)$ is a prewellordering by

$$PRWO(X) :\Leftrightarrow PRO(X) \wedge WF(\prec_X)$$

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We can then say $(D_{\preceq_X}, \preceq_X, \prec_X)$ is a prewellordering by

$$PRWO(X) :\Leftrightarrow PRO(X) \wedge WF(\prec_X)$$

In other words, a prewellordering is a well-founded preordering, aka a transitive, total, well-founded binary relation.

Formalizing (FIX)

Lastly, to say that the diagonalization of a prewellordering is the fixed-point of an operator defined by $F(X, x)$, we write

$$\begin{aligned} FXP_F(X) :\Leftrightarrow\ & PRWO(X) \wedge \\ & \forall x \forall y (x \preceq_X y \leftrightarrow x \prec_X y \vee F(\{z \mid z \prec_X y\}, x)) \end{aligned}$$

If $FXP_F(X)$ holds, we will also write

$$(\Phi^F, \preceq_F, \prec_F)$$

instead of

$$(D_{\preceq_X}, \preceq_X, \prec_X)$$

The Theory $(\mathrm{ACA})_0$

Of its own interest in reverse mathematics, the second-order theory of arithmetical comprehension, $(\mathrm{ACA})_0$ comprises:

- All axioms of NT
- The second-order axiom $(\mathrm{Ind})^2$ of induction:

$$\forall X(\mathbf{0} \in X \wedge \forall x(x \in X \to x + \mathbf{1} \in X) \to \forall x(x \in X))$$

- The scheme $(\Delta^1_0 - \mathrm{CA})$ of arithmetical comprehension:

$$\exists X \forall x(x \in X \leftrightarrow F(x))$$

where $F(x)$ is a first-order formula.

The Theory $(\Pi^0_1 - \mathrm{FXP})_0$

The theory $(\Pi^0_1 - \mathrm{FXP})_0$ contains:

- All axioms of $(\mathrm{ACA})_0$
- For every Π^0_1-formula $F(X, x)$ with X the only free set variable, the axiom

$$\exists X(FXP_F(X)) \qquad \text{(FXP(F))}$$

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Exercise. Prove that $(\Pi^0_1 - \mathrm{FXP})_0 \vdash \exists! X(FXP_F(X))$.
Hint: trivially proves existence. Formalize proof of Prewellordering Theorem to show uniqueness.

Idea and $(\Pi^1_1 - \text{CA})^-$

The theory $(\Pi^1_1 - \text{CA})^-_0$ is a second-order theory with nonlogical axioms:

- All axioms of $(\text{ACA})_0$
- Parameter free Π^1_1-comprehension: for $F(x)$ Π^1_1 with no free set parameters,

$$\exists X \forall x (x \in X \leftrightarrow F(x)) \qquad (\Pi^1_1 - CA^-)$$

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Denoting subtheory by $\sqsubseteq$, we will show

$$\mathrm{ID}_1 \sqsubseteq (\Pi^1_1 - \mathrm{CA})^-_0 \sqsubseteq (\Pi^0_1 - \mathrm{FXP})_0$$

whence

$$||\mathrm{ID}_1|| \leq ||(\Pi^1_1 - \mathrm{CA})^-_0|| \leq ||(\Pi^0_1 - \mathrm{FXP})_0||$$

Embedding the Language

Let $F(X, \vec{x})$ be X-positive arithmetical formula with all free variables shown. The set

$$\mathcal{I}(F) := \{\vec{x} \mid \forall X(\forall \vec{y}[F(X, \vec{y}) \to \vec{y} \in X] \to \vec{x} \in X\}$$

is a set by $\Pi^1_1 - CA^-$.
Therefore, by replacing all occurrences of I_F by $\mathcal{I}(F)$, we can embed $\mathcal{L}(\mathrm{ID})$ in the language of second-order arithmetic. Denote the translation by $(\cdot)^*$. We have

$$(\mathrm{ID}^1_1)^* := \forall \vec{x}(F(\mathcal{I}(F), \vec{x}) \to \vec{x} \in \mathcal{I}(F))$$
$$(\mathrm{ID}^2_1)^* := \forall \vec{x}(F(G, \vec{x}) \to G(\vec{x})) \to \forall \vec{x}(\vec{x} \in \mathcal{I}(F) \to G(\vec{x}))$$

Proving ID^2_1

Lemma

$(\Pi^1_1 - \mathsf{CA})^-_0 \vdash (\mathsf{ID}^2_1)^*$

Proof.

For X-positive arithmetical formula $F(X, \vec{x})$, define $\mathfrak{M}_F(X) :\Leftrightarrow \forall \vec{x}(F(X, \vec{x}) \to \vec{x} \in X)$. For $G(\vec{x})$ an $\mathcal{L}(\mathsf{ID})$ formula, $S := \{\vec{x} \mid G(\vec{x})\}$ is a set. Then we have, inferring from the definition of $\mathcal{I}(F)$,

$$\vec{x} \in \mathcal{I}(F) \leftrightarrow \forall X(\mathfrak{M}_F(X) \to \vec{x} \in X)$$
$$\mathfrak{M}_F(S) \to \vec{x} \in \mathcal{I}(F) \to \vec{x} \in S$$
$$\forall \vec{y}(F(S, \vec{y}) \to \vec{y} \in S) \to \forall \vec{x}(\vec{x} \in \mathcal{I}(F) \to \vec{x} \in S) \qquad ((ID^2_1)^*)$$

$\square$

Proving ID^1_1

Lemma

$(\Pi^1_1 - \mathrm{CA})^-_0 \vdash \left(\mathrm{ID}^2_1\right)^*$

Proof.

$$\begin{aligned}
&\mathfrak{M}_F(X) \to \forall \vec{x}(\vec{x} \in \mathcal{I}(F) \to \vec{x} \in X) && \\
&\mathfrak{M}_F(X) \to F(\mathcal{I}(F), \vec{x}) \to F(X, \vec{x}) && (X\text{-positivity}) \\
&\mathfrak{M}_F(X) \to F(X, \vec{x}) \to \vec{x} \in X && (\text{defn}) \\
&F(\mathcal{I}(F), \vec{x}) \to \mathfrak{M}_F(X) \to \vec{x} \in X && \\
&F(\mathcal{I}(F), \vec{x}) \to \forall X(\mathfrak{M}_F(X) \to \vec{x} \in X) && \\
&F(\mathcal{I}(F), \vec{x}) \to \vec{x} \in \mathcal{I}(F) && \left(\left(\mathrm{ID}^1_1\right)^*\right)
\end{aligned}$$

$\square$

First Subtheory

Theorem

$\mathrm{ID}_1 \sqsubseteq (\Pi^1_1 - CA)^-_0$

Proof.

All axioms of primitive recursive functions, equality are the same. Similarly, any instance of induction in ID_1 has translation provable in $(\Pi^1_1 - CA)^-_0$. The rest follows from the previous two lemmas. □

Final Embedding: Basic Idea

We will show that $(\Pi^1_1 - \mathrm{CA})^-_0 \sqsubseteq (\Pi^0_1 - \mathrm{FXP})_0$ using facts from Chapters 5 and 6, roughly as follows:

1. $(\Pi^0_1 - \mathrm{FXP})_0$ captures positive Π^0_1-definable inductive definitions
2. A Π^1_1-formula with no free set parameters holds iff its search tree is well-founded
3. Those facts can be proved in $(\Pi^0_1 - \mathrm{FXP})_0$
4. This search tree is well-founded iff $\langle\rangle \in \Phi$ where Φ is a positive Π^0_1-definable inductive set
5. Use this to show $(\Pi^0_1 - \mathrm{FXP})_0 \vdash \Pi^1_1 - CA^-$

Positive Π^0_1 Inductive Definitions

Lemma

Let $F(X,x)$ be an X-positive Π^0_1-formula and write $\Phi^F := \{x \mid x \preceq_F x\}$. Then

$$(\Pi^0_1 - \mathrm{FXP})_0 \vdash F(\Phi^F, x) \to x \in \Phi^F$$

$$(\Pi^0_1 - \mathrm{FXP})_0 \vdash \forall y(F(\{z \mid G(z)\}, y) \to G(y)) \to \Phi^F \subseteq \{z \mid G(z)\}$$

Proof.

Straightforward from $FXP_F(X)$ and X-positivity of F. □

Recalling Results

Lemma (Lemma 5.4.7)

If $S^{\omega}_{\langle\Delta\rangle}$ is well-founded, then $\overset{otyp(s)}{\models\!\!=}\delta(s)$ for all $s \in S^{\omega}_{\langle\Delta\rangle}$.

Lemma (Lemma 5.4.8)

If $S^{\omega}_{\langle\Delta\rangle}$ is not well-founded then there are $S_1, \ldots, S_n$ such that $\mathbb{N} \not\models F[S_1, \ldots, S_n]$ for every $F \in \Delta$.

Theorem (Theorem 5.4.9)

For all Π^1_1-sentences of the form $\forall X_1 \cdots \forall X_n F(X_1, \ldots, X_n)$,

$$\mathbb{N} \models \forall X_1 \cdots \forall X_n F(X_1, \ldots, X_n) \Leftrightarrow \exists \alpha < \omega_1^{CK} \overset{\alpha}{\models\!\!=} F(X_1, \ldots, X_n)$$

Proving $(\Pi^1_1 - \mathrm{CA})^-_0 \sqsubseteq (\Pi^0_1 - \mathrm{FXP})_0$

Recall for an arithmetically defined tree T, I_T is the fixed point of

$$F_T(X, x) :\Leftrightarrow T(x) \wedge \forall z(T(x \frown \langle z \rangle) \rightarrow x \frown \langle z \rangle \in X$$

Theorem (Theorem 6.5.5)

An arithmetically definable tree T is well-founded iff $\langle\rangle \in I_T$.

Proving $(\Pi^1_1 - \mathrm{CA})^-_0 \sqsubseteq (\Pi^0_1 - \mathrm{FXP})_0$

Recall for an arithmetically defined tree T, I_T is the fixed point of

$$F_T(X, x) :\Leftrightarrow T(x) \wedge \forall z(T(x \frown \langle z \rangle) \rightarrow x \frown \langle z \rangle \in X$$

Theorem (Theorem 6.5.5)

An arithmetically definable tree T is well-founded iff $\langle\rangle \in I_T$.

Note. Lemmas 5.4.7, 5.4.8, and Theorem 6.5.5 can be proved in $(\Pi^0_1 - \mathrm{FXP})_0$. For the first two, replace induction on $otyp(s)$ by bar induction.

Proving $(\Pi^1_1 - \text{CA})^-_0 \sqsubseteq (\Pi^0_1 - \text{FXP})_0$

Recall for an arithmetically defined tree T, I_T is the fixed point of

$$F_T(X, x) :\Leftrightarrow T(x) \wedge \forall z (T(x \frown \langle z \rangle) \rightarrow x \frown \langle z \rangle \in X$$

Theorem (Theorem 6.5.5)

An arithmetically definable tree T is well-founded iff $\langle\rangle \in I_T$.

Note. Lemmas 5.4.7, 5.4.8, and Theorem 6.5.5 can be proved in $(\Pi^0_1 - \text{FXP})_0$. For the first two, replace induction on $otyp(s)$ by bar induction.

Theorem

$(\Pi^0_1 - \text{FXP})_0 \vdash \Pi^1_1 - CA^-$.

Proving $(\Pi^1_1 - \mathrm{CA})^-_0 \sqsubseteq (\Pi^0_1 - \mathrm{FXP})_0$

Proof.

Let $\forall X F(X, x)$ be Π^1_1 with no other free set parameters. By lemmas 5.4.7 and 5.4.8, $\forall X F(X, n)$ iff $S^\omega_{F(X,n)}$ is well-founded. This search tree is definable by a primitive-recursive formula. Then $G(Y, y, n) :\Leftrightarrow \forall z(y \frown \langle z\rangle \in S^\omega_{F(X,n)} \to y \frown \langle z\rangle \in Y)$ is thus a Y-positive Π^0_1 formula. By Theorem 6.5.5, $S^\omega_{F(X,n)}$ is well-founded iff $\langle\rangle \in \Phi^{G(n)}$. Then $\left\{x \mid \langle\rangle \in \Phi^{G(x)}\right\}$ is a set by arithmetical comprehension, whence

$$(\Pi^0_1 - \mathrm{FXP})_0 \vdash \forall X(F(X, n)) \leftrightarrow \langle\rangle \in \Phi^{G(n)}$$

$$(\Pi^0_1 - \mathrm{FXP})_0 \vdash \exists Z \forall x(x \in Z \leftrightarrow \forall X(F(X, x))) \qquad (\Pi^1_1 - CA^-)$$

$\square$

The Lower Bound

Therefore, we have

$$\psi(\varepsilon_{\Omega+1}) = ||\mathsf{ID}_1|| \leq ||(\Pi^1_1 - \mathsf{CA})^-_0|| \leq ||(\Pi^0_1 - \mathsf{FXP})_0||$$

The Lower Bound

Therefore, we have

$$\psi(\varepsilon_{\Omega+1}) = ||\mathsf{ID}_1|| \leq ||(\Pi^1_1 - \mathsf{CA})^-_0|| \leq ||(\Pi^0_1 - \mathsf{FXP})_0||$$

By taking a detour through $(\Pi_2 - \mathsf{REF})^2$, a second-order theory of Π_2 reflection, we can show that this bound is in fact exact.

References I

[Poh08] Wolfram Pohlers. Ordinal analysis of non-monotone Π^0_1-definable inductive definitions. *Annals of Pure and Applied Logic*, 156(1):160–169, 2008.

[Poh09] Wolfram Pohlers. *Proof Theory: The First Step into Impredicativity*. Springer-Verlag, Berlin, 2009.

[RA74] Wayne Richter and Peter Aczel. Inductive Definitions and Reflecting Properties of Admissible Ordinals. In JE Fenstad and PG Hinman, editors, *Generalized Recursion Theory I, (no 79 in Studies in Logic and the Foundations of Mathematics)*, pages 301–381. North-Holland Publishing Company, Amsterdam, 1974.

Thank You

Questions?