Introduction to Linear Logic

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Structural Motivations

Introduced by Jean-Yves Girard in 1987 [Gir87].
Linear logic is:

- Sequent calculus without weakening and contraction.
- As (or more) constructive than intuitionistic logic, while maintaining desirable features of classical logic.
- Finding more and more applications in theoretical computer science.

High-Level Motivations

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In the conclusion, A still holds. This is perfectly well-suited to mathematics, which deals with stable truths. “But wrong in real life, since real implication is *causal*.”
For beautiful connections with physics, see Baez and Stay 2011 “Physics, Topology, Logic, Computation: a Rosetta Stone” [BS11].

High-Level Motivations

In linear logic, we do not have

$$A \multimap A \otimes A$$

By eliminating weakening and contraction, we eliminate free duplication and elimination of formulas. (We will develop tools to restore these in a controlled manner.)

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By eliminating weakening and contraction, we eliminate free duplication and elimination of formulas. (We will develop tools to restore these in a controlled manner.)
This motivates thinking of formulas in linear logic as *resources* as opposed to eternally true/false propositions. For instance [Gir89, p. 74]:

- state of a Turing machine
- state of a chess game
- chemical solution before/after reaction

Two Sequent Calculi

Consider a standard sequent calculus. Call these "M"-rules:

$$(\mathrm{LM}\wedge)\ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad (\mathrm{LM}\vee)\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'}$$

$$(\mathrm{RM}\wedge)\ \frac{\Gamma \vdash \Delta, A \qquad \Gamma' \vdash \Delta', B}{\Gamma, \Gamma' \vdash \Delta, \Delta', A \wedge B} \qquad (\mathrm{RM}\vee)\ \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}$$

Table: "M"-rules for sequent calculus.

Two Sequent Calculi

Consider a standard sequent calculus. Call these “A”-rules:

$$(\mathrm{LA}\wedge\text{-}1)\ \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad (\mathrm{LA}\vee)\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}$$

$$(\mathrm{LA}\wedge\text{-}2)\ \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad (\mathrm{RA}\vee\text{-}1)\ \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}$$

$$(\mathrm{RA}\wedge)\ \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \qquad (\mathrm{RA}\vee\text{-}2)\ \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}$$

Table: “A”-rules for sequent calculus.

Interderivability of "M" and "A" Rules

In both intuitionistic and classical logic, the two formulations are equivalent.
Here we derive the "M" rules for $\wedge$ using the "A" rules:

$$\dfrac{\dfrac{\Gamma \vdash \Delta, A}{\Gamma, \Gamma' \vdash \Delta, \Delta', A} \qquad \dfrac{\Gamma' \vdash \Delta', B}{\Gamma, \Gamma' \vdash \Delta, \Delta', B}}{\Gamma, \Gamma' \vdash \Delta, \Delta', A \wedge B}\,(\mathrm{RA}\wedge) \qquad \dfrac{\dfrac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B, B \vdash \Delta}\,(\mathrm{LA}\wedge\text{-}1)}{\dfrac{\Gamma, A \wedge B, A \wedge B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}}\,(\mathrm{LA}\wedge\text{-}2)$$

Table: "M" rules derived in "A" system.

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Table: “A” rules derived in “M” system.

Exercise. Carry out the same procedure for the $\vee$ rules.

Interderivability of “M” and “A” Rules

Notice anything?

Interderivability of “M” and “A” Rules

Notice anything?
Every one of those proofs used contraction and/or weakening.
In linear logic, we will have both multiplicative and additive *connectives* corresponding to these two sets of rules which are no longer equivalent.

The Full Language of (Propositional) Classical Linear Logic

- Propositional variables: $A, B, C, \cdots, P, Q, R, \cdots$
- Constants:
  - Multiplicative: $\mathbf{1}, \bot$ (units, resp. of $\otimes, \parr$)
  - Additive: $\top, \mathbf{0}$ (units, resp. of $\&, \oplus$)
- Connectives:
  - Multiplicative: $\otimes, \parr, \multimap$
  - Additive: $\&, \oplus$
- Exponential modalities: $!, ?$
- Linear negation: $(\cdot)^\perp$

Outline I

Outline II

5. Exponentials
   - Exponential Modalities
   - Translation of Intuitionistic Logic
   - Extension of Phase Semantics

6. Conclusion
   - Other Topics
   - References

Sequent Calculus

We now consider the $(\otimes, \multimap, \mathbf{1})$-fragment, **multiplicative intuitionistic linear logic**.

$$(\text{Ax})\ \frac{}{P \vdash P} \qquad (\text{Ex})\ \frac{\Gamma, P, Q, \Delta \vdash C}{\Gamma, Q, P, \Delta \vdash C}$$

$$(\text{Cut})\ \frac{\Gamma \vdash P \qquad P, \Delta \vdash Q}{\Gamma, \Delta \vdash Q}$$

$$(\mathbf{1}\text{-R})\ \frac{}{\vdash \mathbf{1}} \qquad (\mathbf{1}\text{-L})\ \frac{\Gamma \vdash P}{\Gamma, \mathbf{1} \vdash P}$$

$$(\otimes\text{-R})\ \frac{\Gamma \vdash P \qquad \Delta \vdash Q}{\Gamma, \Delta \vdash P \otimes Q} \qquad (\otimes\text{-L})\ \frac{\Gamma, P, Q \vdash R}{\Gamma, P \otimes Q \vdash R}$$

$$(\multimap\text{-R})\ \frac{\Gamma, P \vdash Q}{\Gamma \vdash P \multimap Q} \qquad (\multimap\text{-L})\ \frac{\Gamma \vdash P \qquad Q, \Delta \vdash R}{\Gamma, P \multimap Q, \Delta \vdash R}$$

Table: Sequent Calculus for MILL

Consequences

Theorem

MILL satisfies cut-elimination.

Proof.

Requires defining new commuting conversions, but otherwise is similar to regular intuitionistic case.
See [BBPH93] for a proof (also with !). □

Natural Deduction of MILL

$$\frac{}{P \vdash P}$$

$$(\multimap_{\mathcal{I}})\ \frac{\Gamma, P \vdash Q}{\Gamma \vdash P \multimap Q} \qquad (\multimap_{\mathcal{E}})\ \frac{\Gamma \vdash P \multimap Q \qquad \Delta \vdash Q}{\Gamma, \Delta \vdash Q}$$

$$\frac{}{\vdash I} \qquad (I_{\mathcal{E}})\ \frac{\Gamma \vdash P \qquad \Delta \vdash I}{\Gamma, \Delta \vdash P}$$

$$(\otimes_{\mathcal{I}})\ \frac{\Gamma \vdash P \qquad \Delta \vdash Q}{\Gamma, \Delta \vdash P \otimes Q} \qquad (\otimes_{\mathcal{E}})\ \frac{\Gamma \vdash P \otimes Q \qquad \Delta, P, Q \vdash R}{\Gamma, \Delta \vdash R}$$

Table: Natural Deduction (Sequent Style) for MILL

Term Assignment

$$\overline{x : P \vdash x : P}$$

$$(\multimap_{\mathcal{I}})\ \frac{\Gamma, x : P \vdash f : Q}{\Gamma \vdash \lambda x.f : P \multimap Q} \qquad (\multimap_{\mathcal{E}})\ \frac{\Gamma \vdash f : P \multimap Q \qquad \Delta \vdash g : Q}{\Gamma, \Delta \vdash fg : Q}$$

$$\overline{\vdash * : I} \qquad (I_{\mathcal{E}})\ \frac{\Gamma \vdash f : P \qquad \Delta \vdash g : I}{\Gamma, \Delta \vdash \text{let } g \text{ be } * \text{ in } f : P}$$

$$(\otimes_{\mathcal{I}})\ \frac{\Gamma \vdash f : P \qquad \Delta \vdash g : Q}{\Gamma, \Delta \vdash f \otimes g : P \otimes Q} \qquad (\otimes_{\mathcal{E}})\ \frac{\Gamma \vdash f : P \otimes Q \qquad \Delta, x : P, y : Q \vdash g : R}{\Gamma, \Delta \vdash \text{let } f \text{ be } x \otimes y \text{ in } g : R}$$

Table: Term Assignment for MILL Natural Deduction

Good Features of This Formulation

- Substitution property
- Subject reduction theorem (with commuting conversions added to β)
- Normalization and uniquenesss of normal form

Bad Features of This Formulation

- No subformula property (because of $\otimes$-E)
- Unnecessarily extends term calculus (with let construction)

[Min98] proves a uniqueness of normal form theorem for the $\{\otimes, \&, \multimap\}$ fragment using an extended notion of substitution.

Closed Symmetric Monoidal Categories

In the same way that intuitionistic propositional logic is the logic of Cartesian Closed Categories [Min00, Gol06, TS00], MILL is the logic of **closed symmetric monoidal categories**.

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In the same way that intuitionistic propositional logic is the logic of Cartesian Closed Categories [Min00, Gol06, TS00], MILL is the logic of **closed symmetric monoidal categories**.

I will fly through the relevant definitions; feel free to pursue them when more time is available.

Category

Definition

A category $\mathcal{C}$ is given by a class of objects, $ob(\mathcal{C})$ (we often write $X \in C$ when X is in $ob(\mathcal{C})$) and for every pair of objects X and Y, a set of morphisms, $hom(X, Y)$ (if $f \in hom(X, Y)$, we write $f : X \to Y$). These objects and morphisms must satisfy:

- For each $X \in ob(\mathcal{C})$, $\exists 1_X \in hom(X, X)$.
- Morphisms can be composed: given $f \in hom(X, Y)$ and $g \in hom(Y, Z)$, then $g \circ f \in hom(X, Z)$. (We often write gf for $g \circ f$.)
- If $f \in hom(X, Y)$, then $f 1_X = f = 1_Y f$.
- Composition associates: whenever either is defined, $(hg) f = h (gf)$.

Isomorphism

Definition

A morphism $f \in hom(X, Y)$ is an isomorphism if there is a $g \in hom(Y, X)$ such that $fg = 1_X$ and $gf = 1_y$.

Note that one can provide similar conditions for epi- and mono-morphisms which mirror standard cases of surjections and injections respectively. I only define isomorphisms here because we will see that some inference rules are natural isomorphisms. To understand a natural isomorphism, we must get to the definition of a natural transformation.

Functor

Definition

A functor between categories $\mathcal{C}$ and $\mathcal{D}$, $F : \mathcal{C} \to \mathcal{D}$ sends every $X \in \mathcal{C}$ to $F(X) \in \mathcal{D}$ and every morphism $f \in hom(X, Y)$ to a morphism $F(f) \in hom(F(X), F(Y))$ such that

- For every $X \in \mathcal{C}$, $F(1_X) = 1_{F(X)}$ (i.e. F preserves identity morphisms).
- For every $f \in hom(X, Y)$, $g \in hom(Y, Z)$ in $\mathcal{C}$, $F(gf) = F(g)F(f)$.

Natural Transformation and Isomorphism

Definition

A natural transformation $\eta : F \to G$ between two functors $F, G : \mathcal{C} \to \mathcal{D}$ assigns to every $X \in \mathcal{C}$ a morphism $\eta_X \in hom(F(X), G(X))$ such that for any $f \in hom(X, Y)$, $\eta_Y F(f) = G(f)\eta_X$. That is to say that the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

Definition

A natural isomorphism between functors $F, G : \mathcal{C} \to \mathcal{D}$ is a natural transformation such that η_X is an isomorphism for each $X \in \mathcal{C}$.

Cartesian Product of Categories

Definition

The cartesian product of categories $\mathcal{C}$ and $\mathcal{D}$, denoted by $\mathcal{C} \times \mathcal{D}$, is the category defined as follows:

- Objects are pairs (X, Y) with $X \in \mathcal{C}$ and $Y \in \mathcal{D}$.
- Morphisms in $hom((X, Y), (X', Y'))$ are a pair (f, g) with $f \in hom(X, X')$ and $g \in hom(Y, Y')$.
- Composition is componentwise: $(g, g') \circ (f, f') = (g \circ f, g' \circ f')$.
- Identity morphisms are componentwise: $1_{(X,Y)} = (1_X, 1_Y)$.

Monoidal Category

Definition

A monoidal category is a category $\mathcal{C}$ which also has

- A functor, $\otimes : C \times C \to C$, called the tensor product.
- A unit object $I \in C$
- A natural isomorphism, the associator, which gives isomorphisms for any $X, Y, Z \in \mathcal{C}$

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$$

- Two natural isomorphisms called unitors which assign to each $X \in \mathcal{C}$ isomorphisms

$$l_X : I \otimes X \xrightarrow{\sim} X$$
$$r_X : X \otimes I \xrightarrow{\sim} X$$

Monoidal Category (cont)

Definition

all of which satisfy the following two conditions:

- for every $X, Y \in \mathcal{C}$, the following diagram (the triangle equation) commutes:

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "a_{X,I,Y}"] \arrow[dr, "r_X \otimes 1_Y"'] & & X \otimes (I \otimes Y) \arrow[dl, "1_X \otimes l_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

Monoidal Category (cont)

Definition

- for every $W, X, Y, Z \in \mathcal{C}$, the following diagram (the pentagon equation) commutes:

$$\begin{tikzcd}
& ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "a_{W \otimes X, Y, Z}"'] \arrow[dr, "a_{W,X,Y} \otimes 1_Z"] & \\
(W \otimes X) \otimes (Y \otimes Z) \arrow[ddr, "a_{W,X,Y \otimes Z}"'] & & (W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W, X \otimes Y, Z}"] \\
& & W \otimes ((X \otimes Y) \otimes Z) \arrow[dl, "1_W \otimes a_{X,Y,Z}"] \\
& W \otimes (X \otimes (Y \otimes Z)) &
\end{tikzcd}$$

Braided Monoidal Category

Definition

A braided monoidal category is a monoidal category $\mathcal{C}$ which also has a natural isomorphism (called the braiding) which assigns to every $X, Y \in \mathcal{C}$ an isomorphism

$$b_{X,Y} : X \otimes Y \to Y \otimes X$$

such that the following two diagrams (the hexagon equations) commute:

Braided Monoidal Categories (cont)

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "a^{-1}_{X,Y,Z}"] \arrow[d, "b_{X,Y\otimes Z}"'] & (X \otimes Y) \otimes Z \arrow[r, "b_{X,Y}\otimes 1_Z"] & (Y \otimes X) \otimes Z \arrow[d, "a_{Y,X,Z}"] \\
(Y \otimes Z) \otimes X & Y \otimes (Z \otimes X) \arrow[l, "a^{-1}_{Y,Z,X}"] & Y \otimes (X \otimes Z) \arrow[l, "1_Y \otimes b_{X,Z}"]
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "a_{X,Y,Z}"] \arrow[d, "b_{X\otimes Y,Z}"'] & X \otimes (Y \otimes Z) \arrow[r, "1_X\otimes b_{Y,Z}"] & X \otimes (Z \otimes Y) \arrow[d, "a^{-1}_{X,Z,Y}"] \\
Z \otimes (X \otimes Y) & (Z \otimes X) \otimes Y \arrow[l, "a_{Z,Y,X}"] & (X \otimes Z) \otimes Y \arrow[l, "b_{X,Z}\otimes 1_Y"]
\end{tikzcd}$$

Symmetric Monoidal Category

Definition

A symmetric monoidal category is a braided monoidal category $\mathcal{C}$ such that for every $X, Y \in \mathcal{C}$, $b_{X,Y} = b^{-1}_{Y,X}$.

Closed Symmetric Monoidal Category

Definition

A closed symmetric monoidal category is a symmetric monoidal category $\mathcal{C}$ with, for any two objects $X, Y \in \mathcal{C}$,

- an object $X \multimap Y$
- a morphism $app_{X,Y} : X \otimes (X \multimap Y) \to Y$

which satisfies a universal property: for every morphism $f : X \otimes Z \to Y$, there exists a unique morphism $\lambda_Z^{X,Y} : Z \to (X \multimap Y)$ such that $f = app_{X,Y} \circ (1_X \otimes \lambda_Z^{X,Y})$, i.e. such that the following diagram commutes:

$$\begin{tikzcd} X \otimes Z \arrow[rr, "1_X \otimes \lambda_Z^{X,Y}"] \arrow[dr, "f"'] & & X \otimes (X \multimap Y) \arrow[dl, "app_{X,Y}"] \\ & Y & \end{tikzcd}$$

Soundness and Completeness

Theorem

For any closed symmetric monoidal category $\mathcal{C}$, there is an interpretation function

$$[\![\cdot]\!] : \mathcal{L}_{MILL} \to \mathcal{C}$$

such that $\Gamma \vdash_{MILL} A$ *iff there is a morphism* $t : [\![\Gamma]\!] \to [\![A]\!]$ *in* $\mathcal{C}$.

Linear Negation

Linear negation, $(\cdot)^{\perp}$, is involutive and defined by De Morgan equations:

$$
\begin{aligned}
\mathbf{1}^{\perp} &:= \perp \\
\perp &:= \mathbf{1} \\
\left(p^{\perp}\right)^{\perp} &:= p \\
(P \otimes Q)^{\perp} &:= P^{\perp} \parr Q^{\perp} \\
(P \parr Q)^{\perp} &:= P^{\perp} \otimes Q^{\perp}
\end{aligned}
$$

Note: $p^{\perp}$ is now considered atomic. Linear implication is a defined connective:

$$P \multimap Q := P^{\perp} \parr Q$$

One-Sided Sequent Calculus

With linear negation, we may consider calculi with no formulas on the left of $\vdash$. For each subsystem, one can show that $\Gamma \vdash \Delta$ iff $\vdash \Gamma^{\perp}, \Delta$.[1]

$$\vdash P^{\perp}, P \qquad \frac{\vdash \Gamma, P \qquad \vdash P^{\perp}, \Delta}{\vdash \Gamma, \Delta}$$

$$\vdash \mathbf{1} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}$$

$$\frac{\vdash \Gamma, P \qquad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q} \qquad \frac{\vdash \Gamma, P, Q}{\vdash \Gamma, P \parr Q}$$

Table: Sequent Calculus for MLL

[1] For a two-sided sequent calculus of the full first-order classical linear logic, see [TS00, p. 294-295].

Motivation

Limitations of natural deduction [Gir95]:

1. Cannot handle symmetry (desire multiple conclusions)
2. Rules with assumption discharge (i.e. $\multimap$-I) apply to whole proofs, not formulas
3. Our $\otimes$-E rule requires commuting conversions just like $\forall$-E does in NJ; these conversions are cumbersome

Girard develops a new notation, proof nets, to avoid these worries. First, we focus on just the $(\otimes, \parr)$-fragment, ignoring constants.

Proof Structures

Definition

A proof structure consists of

1. Occurrences of formulas, A_i
2. Links between said occurrences, of three kinds:
    1. Axiom links
    $$\frac{}{P_i \qquad P_j^{\perp}}$$
    2. Times link:
    $$\frac{P_i \qquad Q_j}{(P \otimes Q)_k}$$
    Here, P_i and Q_j are premises and $(P \otimes Q)_k$ is a conclusion.
    3. Par link:
    $$\frac{P_i \qquad Q_j}{(P \parr Q)_k}$$
    Here, P_i and Q_j are premises and $(P \parr Q)_k$ is a conclusion.

Proof Structures

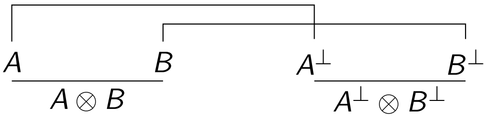
Definition

such that

1. every occurrence of a formula is the conclusion of exactly one link
2. every occurrence of a formula is the premise of at most one link

Need for a Criterion of Correctness

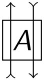
The idea is that a proof structure with conclusions $A_1, \ldots, A_n$ in fact proves $A_1 \parr \cdots \parr A_n$.
As defined, proof structures can be well-formed even if the associated $\parr$ is not provable.



To establish a criterion of correctness, we first introduce the notion of a *trip*.
(This is Girard's original criterion. See [DR89] for an alternative with lower computational complexity.)

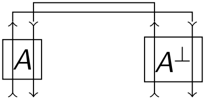
Links and Time

We now view each formula as a box through which a particle can travel:



The two operations of entering and exiting A along the same arrowed path are performed in the same unit of time, $t^{\uparrow}$ or $t_{\downarrow}$. At $t^{\uparrow}$, the particle is between the two upward arrows and nowhere else. We must reformulate the notion of proof structure to accommodate this picture.

Trips: Axiom Link



$$t\left(A^{\perp}_{\downarrow}\right) = t\left(A^{\uparrow}\right) + 1$$
$$t\left(A_{\downarrow}\right) = t\left(A^{\perp\uparrow}\right) + 1$$

Trips: Terminal Formula

$$t\left(A^{\uparrow}\right) = t\left(A_{\downarrow}\right) + 1$$

Trips: Times Link

"L"	"R"
$t\left(B^{\uparrow}\right) = t\left(A \otimes B^{\uparrow}\right) + 1$	$t\left(A^{\uparrow}\right) = t\left(A \otimes B^{\uparrow}\right) + 1$
$t\left(A^{\uparrow}\right) = t\left(B_{\downarrow}\right) + 1$	$t\left(B^{\uparrow}\right) = t\left(A_{\downarrow}\right) + 1$
$t\left(A \otimes B_{\downarrow}\right) = t\left(A_{\downarrow}\right) + 1$	$t\left(A \otimes B_{\downarrow}\right) = t\left(B_{\downarrow}\right) + 1$

Table: Time Equations for Two Switches of Times Link

Trips: Par Link

"L"	"R"
$t\left(A^{\uparrow}\right) = t\left(A ⅋ B^{\uparrow}\right) + 1$	$t\left(B^{\uparrow}\right) = t\left(A ⅋ B^{\uparrow}\right) + 1$
$t\left(A ⅋ B_{\downarrow}\right) = t\left(A_{\downarrow}\right) + 1$	$t\left(A ⅋ B_{\downarrow}\right) = t\left(A_{\downarrow}\right) + 1$
$t\left(B^{\uparrow}\right) = t\left(B_{\downarrow}\right) + 1$	$t\left(A^{\uparrow}\right) = t\left(A_{\downarrow}\right) + 1$

Table: Time Equations for Two Switches of Par Link

Short vs. Long Trips

Set switches arbitrarily. Pick an arbitrary formula and exit gate at $t = 0$. By construction, there are clear, unambiguous directions on how to proceed indefinitely.
Because this is a finite structure, however, every trip is periodic. Let k be the smallest positive integer such that the particle inters through the gate from which it left at $t = 0$. Denoting by p the number of formulas in the structure, we call a trip

- short, if $k < 2p$
- long, if $k = 2p$

Two examples, on board.

Proof Net Defined

Definition

A proof net is a proof structure which admits no short trip.

Equivalently, but slightly more formally:

Definition

A proof net is a proof structure with p formulas (and n switches, a set E of exits) such that for any position of the switches, there is a bijection

$$t : \mathbb{Z}/2p\mathbb{Z} \to E$$

such that for any $e, e' \in E$, $t(e') = t(e) + 1$ iff e' immediately follows e in the travel process.

A Net for Every Proof

Theorem

If π is a proof $\vdash A_1, \ldots, A_n$ in the sequent calculus of multiplicative linear logic without exponentials, constants, and cut, then there is a proof-net π^- whose terminal formulas are exactly one occurrence each of $A_1, \ldots, A_n$.

Proof

Base case: $\pi = \vdash A, A^\perp$. Trivially, take π^- to be the proof-net

$$\overline{A \qquad A^\perp}$$

Case 1: π is obtained from λ by exchange rule. Take $\pi^- = \lambda^-$.

A Net for Every Proof

Proof (cont)

Case 2: π is

$$\dfrac{\dfrac{\lambda}{\vdash A, B, C}}{\vdash A \parr B, C}$$

Let π^- be the structure (invoking the inductive hypothesis)

$$\begin{array}{c}\lambda^- \\ \dfrac{A \qquad B}{A \parr B}\end{array}$$

π^- is a net: set all switches of λ^- arbitrarily and assume (WLOG) new link is on "L". By IH, λ^- is a sound net with n swithces. At $t = 2n - 1$, arrive at $A_\downarrow$. Travelling through $A \parr B_\downarrow$, $A \parr B^\uparrow$ at $t = 2n, 2n + 1$ yields a long trip.

A Net for Every Proof

Proof.

Proof (cont) Case 3: π is

$$\dfrac{\begin{array}{cc} \lambda & \mu \\ \vdash A, C & \vdash B, D \end{array}}{\vdash C, D, A \otimes B}$$

Let π^- be the structure

$$\dfrac{\begin{array}{cc} \lambda^- & \mu^- \\ A & B \end{array}}{A \otimes B}$$

Assume λ^- has n formulas, and μ^- m. Starting at $A^\uparrow$ at $t = 0$, one arrives at $A_\downarrow$ at $2n - 1$. Then $t\left(B^\uparrow\right) = 2n$. Since μ^- is sound (IH), $t\left(B_\downarrow\right) = 2n + 2m - 1$. Then, travelling through $A \otimes B_\downarrow$ and $A \otimes B^\uparrow$ at $2n + 2m, 2n + 2m + 1$ yields a long trip. $\square$

Not Injective

Theorem

The map $(\cdot)^-$ from proofs to proof nets is not injective.

Proof.

The two proofs below are distinct but mapped to the same net.

$$\dfrac{\dfrac{\dfrac{\vdash A, A^\perp \qquad \vdash B, B^\perp}{\vdash A^\perp, B^\perp, A \otimes B}}{\vdash A^\perp \parr B^\perp, A \otimes B} \qquad \vdash C, C^\perp}{\vdash A^\perp \parr C^\perp, C^\perp, (A \otimes B) \otimes C}
\qquad\qquad
\dfrac{\dfrac{\dfrac{\vdash A, A^\perp \qquad \vdash B, B^\perp}{\vdash A^\perp, B^\perp, A \otimes B} \qquad \vdash C, C^\perp}{\vdash A^\perp, B^\perp, C^\perp, (A \otimes B) \otimes C}}{\vdash A^\perp \parr B^\perp, C^\perp, (A \otimes B) \otimes C}$$

Table: Two Distinct Proofs With Same Net

$\square$

A Proof for Every Net

Theorem

For every proof-net β, there is a sequent calculus proof π such that $\beta = \pi^{-}$.

Proof

Induction on the number of links in β.
Base case: one link. π is an axiom.

A Proof for Every Net

Proof (cont)

Case 1: β has more than one link. Assume β has a terminal formula which is the conclusion of a *par* link:

$$\begin{array}{c} \beta' \\ \dfrac{A \qquad B}{A \parr B} \end{array}$$

Because β is a proof-net, so too is β' (exercise). By IH, there is a proof π' such that $\beta' = \pi'^{-}$. Then let π be:

$$\begin{array}{c} \pi' \\ \dfrac{\vdash A, B}{\vdash A \parr B} \end{array}$$

A Proof for Every Net

Proof.

Case 2: β has more than one link, but no terminal formula is the conclusion of a *par* link.
This case is surprisingly subtle and much more complex than the previous case.
See [Gir87, p. 35-40] for the details. □

What About Cut?

Define a cut-link in a proof structure as:

$$\frac{A \qquad A^{\perp}}{\mathrm{CUT}}$$

In what follows, let β be a proof-net containing a CUT link. We define a *contractum* β' as follows.

Contraction

If β ends

$$\dfrac{\dfrac{\begin{matrix}\vdots & \qquad & \vdots\\ B & & C\end{matrix}}{BmC} \qquad \dfrac{\begin{matrix}\vdots & \qquad & \vdots\\ B^{\perp} & & C^{\perp}\end{matrix}}{B^{\perp}m^{\perp}C^{\perp}}}{\mathrm{CUT}}$$

where $m, m^{\perp}$ are dual multiplicatives, β' has this part replaced with

$$\dfrac{\begin{matrix}\vdots & \qquad & \vdots\\ B & & B^{\perp}\end{matrix}}{\mathrm{CUT}} \qquad \dfrac{\begin{matrix}\vdots & \qquad & \vdots\\ C & & C^{\perp}\end{matrix}}{\mathrm{CUT}}$$

Contraction

If A is conclusion of an axiom link, unify the $A^\perp$ in the axiom with the $A^\perp$ in the CUT:

$$\begin{array}{c} \vdots \\ A^\perp \\ \vdots \end{array}$$

Same for when $A^\perp$ conclusion of an axiom link. If both are conclusions of different axiom links, contract to

$$\begin{array}{cc} \hline A & A^\perp \\ \vdots & \vdots \end{array}$$

Cut Elimination

Write β red β' if β' results from one or more contractions of β.
A few results (see [Gir87, p. 42-43] for proofs):

Theorem

1. *If β is a proof-net and β red β', then β' is a proof-net.*
2. *If β red β', β is strictly larger than β' (in terms of number of formulas).*
3. *Church-Rosser property: If β red β' and β red β'', there exists β''' such that β' red β''' and β'' red β'''.*
4. *Strong Normalization: A proof-net of size n normalizes into a cut-free proof-net in less than n steps.*

Additive Connectives

We introduce the additive connectives $\&, \oplus$ with units $\top, \mathbf{0}$ respectively.

$$(P \,\&\, Q)^{\perp} := P^{\perp} \oplus Q^{\perp}$$
$$(P \oplus Q)^{\perp} := P^{\perp} \,\&\, Q^{\perp}$$

$$\vdash \Gamma, \top \qquad \text{no rule for } \mathbf{0}$$

$$\frac{\vdash \Gamma, P \qquad \vdash \Gamma, Q}{\vdash \Gamma, P \,\&\, Q}$$

$$\frac{\vdash \Gamma, P}{\vdash \Gamma, P \oplus Q} \qquad \frac{\vdash \Gamma, Q}{\vdash \Gamma, P \oplus Q}$$

Table: Sequent Calculus Rules for Additives

Intuition Behind Additives

These correspond to the additive formulation of the connectives given in the introduction.

Because of common context, & ("with") is something like a superposition.

Consider a metaphor: I have \$1 (call this P) and am at a vending machine which has both a candy bar (Q) and a bag of chips (R) each for sale for \$1.

I have $P \multimap Q$ and $P \multimap R$, but not $P \multimap Q \otimes R$ since this combination would require \$2. But, I do have $P \multimap Q \,\&\, R$. This says I can get either a candy bar or a bag of chips, but not both, with my dollar.

Extending Proof Nets

Given the beautiful picture of proof nets that we just saw, it's natural to want to extend them to include the additives. This, however, is not a trivial task and gave Girard a lot of trouble. [HvG05] has developed proof-nets for the multiplicative-additive fragment without exponentials or units.
Because this development is quite complex and different from the nets we developed for the multiplicatives, I will only sketch the approach.

Extending Proof Nets

1. For MLL, inductively define a “linking” on a sequent. The corresponding graph will be a proof-net if all $\parr$-switchings are trees.
2. Extend definition of linking to MALL.
   - Using notion of “additive resolution”: delete one argument subtree from each additive connective.
   - Each additive resolution induces an MLL proof structure.
3. Associate with each sequent a *set* of linkings.
4. Two more notions: toggling, switching cycle
5. A set θ of linkings on $\vdash \Gamma$ is a MALL proof-net iff:
   1. Exactly one $\lambda \in \theta$ is on each additive resolution
   2. Each $\lambda \in \theta$ induces an MLL net.
   3. Every set Λ of ≥ 2 linkings toggles a & that is not in any switching cycle.

Phase Semantics

I will introduce a basic semantics in terms of phase spaces. There is a more complex semantics in terms of *coherent spaces* that would take too long to develop in this talk.

Definition

A *phase space* $(P, \perp_P)$ consists of:

1. a commutative monoid P (an abelian group without inverse property)
2. a set $\perp_P \subseteq P$ called the *antiphases* of P.

Facts

Definition

For every $G \subseteq P$, we define

$$G^{\perp} := \{p \in P \mid \forall q \in G, pq \in \perp_P\}$$

Definition

A set $G \subseteq P$ is a *fact* if $G^{\perp\perp} = G$. The elements of a fact G are called *phases*. A fact G is *valid* when $1 \in G$.

Proposition

G is a fact iff $G = H^{\perp}$ for some $H \subseteq P$.

Examples of Facts

Examples

1. $\bot = \{1\}^\perp$ is a fact.
2. $\mathbf{1} := \bot^\perp$ is a submonoid.
3. $\top := \emptyset^\perp = P$
4. $\mathbf{0} := \top^\perp$ is the smallest fact.

Closure Under Intersection

Theorem

Facts are closed under arbitrary intersection.

Proof.

Let $(G_i)_{i \in I}$ be a family of facts. We show that $\cap_i G_i = \left(\cup G_i^{\perp}\right)^{\perp}$ which is a fact by the previous proposition.

$\cap G_i \subseteq \left(\cup G_i^{\perp}\right)^{\perp}$: Suppose $g \in \cap G_i = \cap G_i^{\perp\perp}$. Let $q \in \cup G_i^{\perp}$. For some $i_0 \in I$, $q \in G_{i_0}^{\perp}$. But $g \in G_{i_0}^{\perp\perp}$, so $gq \in \perp$.

$\left(\cup G_i^{\perp}\right)^{\perp} \subseteq \cap G_i$: Suppose $g \notin \cap G_i$. Then for some i_0, $g \notin G_{i_0} = G_{i_0}^{\perp\perp}$. Therefore, $\exists q \in G_{i_0}^{\perp}$ such that $gq \notin \perp$. But we also have $q \in \cup G_i^{\perp}$, and so $g \notin \left(\cup G_i^{\perp}\right)^{\perp}$. Take contrapositive. $\square$

Definition of Connectives

First, we define the product of subsets. For any $G, H \subseteq P$,

$$G \cdot H := \{gh \in P \mid g \in G, h \in H\}$$

From here out, suppose G and H are facts.

Definition

The "connectives" are defined as follows:

1. $G \multimap H = \{p \in P \mid \forall g \in G, pg \in H\}$
2. $G \otimes H = (G \cdot H)^{\perp\perp}$
3. $G \parr H = \left(G^{\perp} \cdot H^{\perp}\right)^{\perp}$
4. $G \,\&\, H = G \cap H$
5. $G \oplus H = (G \cup H)^{\perp\perp}$

Properties

The facts $\mathbf{1}, \perp, \mathbf{0}, \top$ are units of the operations on facts $\otimes, ⅋, \oplus, \&$ respectively.
The three multiplicatives can be defined from any one of them plus $(\cdot)^{\perp}$.
For a whole host of other properties (such as distribution, etc), see [Gir87, p. 19-21].

Phase Structures

Definition

A phase structure for the language of propositional linear logic is a phase space $(P, \perp_P)$ with a function s that maps each propositional letter p to a fact $s(p)$ of P.

An interpretation function S from the full language of propositional linear logic to facts is defined in the obvious way: associate with each connective the equivalent operation on facts. We then say:

Definition

1. A is valid in S when $1 \in S(A)$.
2. A is a linear tautology when A is valid in any phase structure.

Soundness and Completeness

Theorem

The sequent calculus of MALL is sound and complete with respect to phase semantics.

Proof.

Soundness: interpret $\vdash \Gamma$ as $\parr\Gamma$ and do a straightforward induction on the sequent.
Completeness: define $Pr(A) = \{\Gamma \mid \vdash \Gamma, A\}$. Verify: $Pr(A)$ is a fact for every formula A. Define a phase structure as follows: M contains all multisets of formulas (exercise: prove that multisets of formulas form a monoid with concatenation as operation and $\emptyset$ as unit), $\perp_M = \{\Gamma \mid \vdash \Gamma\} = Pr(\perp)$, and $S(a) = Pr(a)$. Verify that $S(A) = Pr(A)$ by induction on A. Now, assume A a linear tautology. Then A is valid in S and so $\emptyset \in S(A) = Pr(A)$, i.e. $\vdash A$. $\square$

Introducing the Exponential Modalities

As defined so far, linear logic is strictly weaker than either intuitionistic or classical logic. To restore the expressive power that was lost by eliminating structural rules, we re-introduce these rules in a controlled manner via the modalities ! ("of course") and ? ("why not").

[These roughly correspond to $\Box$ and $\Diamond$.]

Extend linear negation:

$$(P!)^{\perp} := ?\left(P^{\perp}\right)$$

$$(?P)^{\perp} := !\left(P^{\perp}\right)$$

Extending Sequent Calculus

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A} \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}$$

Table: Sequent Calculus Rules for Exponentials

Think of ! as free duplication of a resource and ? as discarding thereof. Operational semantics of linear logic [Abr93] make the connection with memory management explicit.

Examples

Embedding Intuitionistic Logic in Linear Logic

Define a translation $(\cdot)^*$ from formulas of intuitionistic logic to formulas of linear logic as follows (atomic formulas directly carried over):

$$
\begin{aligned}
(P \to Q)^* &= (!P^*) \multimap Q^* \\
(P \wedge Q)^* &= P^* \,\&\, Q^* \\
(P \vee Q)^* &= !P^* \oplus !Q^* \\
(\neg P)^* &= ?\,(P^*)^{\perp}
\end{aligned}
$$

Then $\Gamma \vdash A$ is provable intuitionistically iff $!\Gamma^* \vdash A^*$ is provable linearly.
Gödel's double-negation translation of classical logic into intuitionistic logic can be composed with this translation to embed classical logic inside linear logic as well.

Phase Semantics for Exponentials

First, define (recalling that $\mathbf{1} = \bot^{\top} = \{1\}^{\bot\bot}$)

$$I := \{p \in \mathbf{1} \mid pp = p\}$$

Then our soundness and completeness results extend by extending the interpretation of formulas by (G is assumed to be a fact)

$$!G := (G \cap I)^{\bot\bot}$$
$$?G := \left(G^{\bot} \cap I\right)^{\bot}$$

Nota bene. Girard originally developed topolinear spaces to accommodate the exponentials. The definition given here appears in [Gir95].

Explore More

Some topics that I did not include that have been well-explored:

- Quantifiers. These don't add much unexpected complexity. Girard is also famous for his System F of second-order propositional logic which underlies the programming language ML; he has developed an analogous version of linear logic.
- Coherent space semantics.

Unrestricted Comprehension and Russell's Paradox

Unrestricted comprehension says, informally, that for any property $\varphi(x)$, we can form the set $\{x \mid \varphi(x)\}$. Russell famously proved a paradox by forming the set

$$R = \{x \mid x \notin x\}$$

It follows that $R \in R \Leftrightarrow R \notin R$.
Two ways to respond:

1. Weaken comprehension. By far the dominant approach. Whence restricted comprehension, the axiom of foundation, and the hierarchical set-theoretic universe.
2. Weaken the underlying logic.

Unrestricted Comprehension in Linear Logic

Mints' student Shirahata [Shi94] pursued the second approach and proved that *unrestricted comprehension is consistent in (various systems of) linear logic*.

Won't go into details here, but notice that a standard proof of one direction of Russell's paradox uses contraction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R \in R \vdash R \in R \qquad \bot \vdash \bot}{R \in R \to \bot, R \in R \vdash \bot}}{R \in R, R \in R \vdash \bot}}{R \in R \vdash \bot}}{\vdash R \in R \to \bot}}{\vdash R \in R}$$

Geometry of Interaction

Three levels of semantics in logic:

$$\begin{aligned} \text{Formulas} &\mapsto \text{model theory} \\ \text{Proofs} &\mapsto \text{denotational semantics} \\ \text{Cut elimination} &\mapsto \text{geometry of interaction} \end{aligned}$$

Basic idea: formulas are spaces, proofs are operators on these spaces, operators interact. Also gives some geometrical intuition to negation as orthogonality.

I personally need to explore this area more.

Chemical Logic

The subject of my undergraduate thesis.
Basic idea: Girard many places mentions analogy with chemistry and writes a chemical formula as

$$H_2 \otimes H_2 \otimes O_2 \multimap H_2O \otimes H_2O$$

My idea: incorporate covalence (sharing of resources) into linear logic so that well-balanced chemical equations are derivable.
Extend language with set of valences $e, f, g, \ldots$, new atomic form $(e, \ldots, e_n)P$, and a connective $\overset{e}{|}$ for every valence item.
Status: no good inference rule candidates (none changed expressive power of the logic). But did develop it more fully than this sketch.
Needs more motivation; any thoughts?

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Thank You

Questions?